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Abstract

Presented is a linear program for matrix algebra required in a first course in multivariate educational statistics. The purpose of the program is to enable graduate students, through self instruction, to acquire sufficient knowledge of matrix algebra to meet the prerequisite of a course in multivariate statistics of a type taught in a department of education. This course introduces the student to multiple and partial correlation, canonical correlation, and multivariate analysis of variance. It assumes some knowledge of matrix operations, determinents, linear dependence and vector spaces, and the characteristic equation of the matrix. A preliminary trial of the program was carried out with 28 graduate students. Analysis of errors made by the students and the reaction of the students to the material was performed for the purpose of revising the program. The revised material was given to another 29 graduate students and necessary revision was made. (RP)

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DEVELOPMENT OF AN INSTRUCTIONAL AID FOR A COURSE IN MULTIVARIATE EDUCATIONAL STATISTICS

Vidya Bhushan

FINAL REPORT

PROJECT NO. 8-I-080

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U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
OFFICE OF EDUCATION

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Vidya Bhushan

Education Research and Development Center
University of Hawaii
Honolulu, Hawaii

July, 1969

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TABLE OF CONTENTS

Page

SUMMARY 1

INTRODUCTION 2

METHODS 3

REFERENCES 4

APPENDIX

Instructions 5

Unit I Definition 6

Unit II Equality of Matrices 18

Unit III Addition of Matrices 25

Unit IV Subtraction of Matrices 40

Unit V Numerical Multiples of Matrices 49

Unit VI Multiplication of Matrices 58

Unit VII Properties of Matrix Multiplication 68

Unit VIII The Laws of Matrix Multiplication 88

Unit IX Powers of Matrices 102

Unit X Determinants 118

Unit XI Inversion of Matrices 135

Unit XII Inversion of Matrices (continued) 164

Unit XIII Inversion of Matrices (other method) 190

Unit XIV Vectors 199

Unit XV The Solution of Linear Equations 223

Unit XVI The Theory of Linear Dependence 229

Unit XVII The Characteristic Equation of a Matrix 238

Unit XVIII Special Types of Matrices 246

SUMMARY

A linear program for matrix algebra required in a first course in multivariate educational statistics was developed. The purpose of the program is to enable graduate students to acquire, through self-instruction, sufficient knowledge of matrix algebra to meet the prerequisite of a course in multivariate statistics of a type taught in a department of education. This course introduces the student to multiple and partial correlation, canonical correlation, and multivariate analysis of variance. It assumes some knowledge of matrix operations, determinants, linear dependence and vector spaces, and the characteristic equation of the matrix. The linear program covers these topics.

Material from standard texts in matrix algebra was incorporated in a linear program. The material was selected in the light of the experience in teaching this subject and the reviews of this material in the multivariate texts.

A preliminary trial of the program was carried out with twenty-eight graduate students from the University of Chicago and the University of Hawaii. Analysis of errors made by the students and the reaction of the students to the material was performed for the purpose of revising the program. The revised material was given to another twenty-nine graduate students of the Department of Education to read, and necessary revision was made.

INTRODUCTION

The sequence of educational statistics courses at many universities includes an introduction to the use of multivariate methods in educational research. The courses cover multiple and partial correlation, canonical correlation, multivariate analysis of variance, and component and factor analysis. Since all textbooks and other literature in the field of multivariate analysis make use of matrix algebra, it is impossible to teach the courses without assuming some knowledge of this subject on the part of the students. Many schools with a theoretically oriented mathematics department have no course in matrix algebra suitable for applied workers. Since it has seemed undesirable to include a purely mathematical subject in the education curriculum, we have had to depend on our students to prepare themselves in matrix algebra by their own resources. Although a number of conventional texts are available for this purpose (Aiken, 1956; Browne, 1958; Horst, 1963; Ayers, 1962), the students appear to need more guidance in approaching the subject than can be offered in a textbook. It was therefore proposed to prepare a linear program, which will lead the student through the material with essentially no errors.

In recent years several studies have been made of the effectiveness of linear programs compared with other methods of teaching. The results are still inconclusive, but the linear programs definitely take less time than other methods to learn the same material. Studies by Hughes and McNamara (1961), Porter (1961), Hough (1962), and Smith (1961) are some of the examples which confirm this fact. Bhushan (1966) also conducted a study of the effectiveness of two methods of teaching elementary matrix algebra and did not find any significant difference between the two, but the errorless program took significantly less time than the dialectical program. This result encourages me to believe that the student will find the program an efficient way to prepare themselves in matrix algebra.

METHODS

The contents of the program were taken from standard texts in matrix algebra (Browne, 1958; Horst, 1963; Hohn, 1958; Kemeny, 1957; Murdock, 1957; Schwartz, 1961) and were incorporated in a linear program. Professor Bock who teaches a course in multivariate analysis at the University of Chicago and multivariate texts of Anderson (1958) and Rao (1965) guided the selection of the material to be included in the program. The contents were divided into eighteen units. The program was revised on the basis of the experts' suggestions.

For the evaluation of the material, it was given to the graduate students of the department of Education, thirteen from the University of Chicago and fifteen from the University of Hawaii, to read. They were asked to read each frame, provide the necessary answer and compare it with the one given just after the frame but masked with a piece of paper. If the answer did not tally with the given answer, they checked it with a red pencil and proceeded further. They were also asked to point out if they had any difficulty in understanding any frame. At the end they were given an achievement test. The mean achievement scores for Groups 1 and 11 were 30.5 and 29.0 which are much above the mean of the normally distributed scores of the achievement test. The maximum possible score for the achievement test was 42. It indicates that both the groups learned matrix algebra very well.

On the basis of their responses, the error rates were counted for each frame. Those frames which had higher error rates (more than 10%) or had ambiguity of language, were rewritten.

Thus revised materials were given to other graduate students of the department of Education, twelve from the University of Chicago and seventeen from the University of Hawaii, to read, and the above procedure for revising the program was repeated. This time the students were asked to note down the total time taken to read the whole material. The average time to read the program is ten hours and thirty minutes. The revised program is given in the appendix.

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APPENDIX

INSTRUCTIONS

Each numbered section in the text is called a "frame" and in most such frames a written response is called for at some point. Following the frame is the "answer" which is the correct answer. Therefore, proceed as follows:

1. Use another sheet of paper to mark the printed "answer" while you read the frame and write your response in the blank space.

2. Move the paper down and compare your response with the printed answer. If they are identical, proceed to the next frame. If they differ, mark your answer with a colored pencil mark, then proceed to the next frame.

3. You can work according to your speed as long as you understand the material well.

Go to the next page and begin your work.

Unit I: Definition

1. A matrix is basically a very simple way of organizing the numbers used to describe the various kinds of information that come to our attention every day. For example the scores of two students in three subjects can be written conveniently as follows:

$$\begin{array}{ll|ccc|} & & \multicolumn{3}{c}{\text{Subjects}} \\ & & \text{Eng} & \text{Math} & \text{Hist} \\ \text{Students} & \text{1st} & 15 & 50 & 31 \\ & \text{2nd} & 14 & 52 & 35 \end{array}$$

The score of the 1st student in History would be 31, and the score of the 2nd student in Math would be 52. Thus a matrix is a rectangular array of elements. The above matrix has 2 rows and 3 columns and it is said to be a 2 x 3 (read 2 by 3) matrix. In the above arrangement, the numbers 15, 50, 31 form the first row and the numbers ..., ..., ... form the second row. The numbers 15, 14 form the first column, numbers 50, 52 form the second column and the numbers ..., ... form the third column.

14, 52, 35

31, 35

---

2. The scores of Arithmetic, English and History of mid term exam of three students are 6, 8, 9; 8, 10, 9; and 7, 8, 8 respectively. Arrange these numbers in the following 3 x 3 matrix.

$$\begin{array}{ll|ccc|} & & \multicolumn{3}{c}{\text{Scores}} \\ & & \text{Arith} & \text{Eng} & \text{Hist} \\ & \text{1st} & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \text{Students} & \text{2nd} & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ & \text{3rd} & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array}$$

$$\begin{bmatrix} 6 & 8 & 9 \\ 8 & 10 & 9 \\ 7 & 8 & 8 \end{bmatrix}$$

3. If a matrix has 3 rows and 4 columns, it is said to be .... x .... matrix.
(number) (number)

3 x 4

4. If a matrix has n rows and m columns, it is said to be .... x .... matrix.

n x m

5. If a matrix happens to have the same number of rows as columns it is called a square matrix. If there are 2 numbers in each row and column of the array, the matrix is said to be a 2 by 2 matrix, or a matrix of size 2. If there are 3 numbers in each row and column of the array, the matrix is said to be a matrix of size ....
(number)

3

---

6. If there are 4 numbers in each row and column of the array, the matrix is said to be a .... by .... matrix or a matrix of size ....
(number) (number) (number)

---

4 by 4

4

---

7. If there are n numbers in each row and column of the array, the matrix is said to be an .... by .... matrix or a matrix of size ....

---

n by n

n

8. The arrays

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$$

are .... x .... matrices, or matrices of size ....

2 x 2

2

9. The array

$$\begin{bmatrix} 2 & 4 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

is a .... x .... matrix, because it has .... rows and ....
(number) (number) (number) (number)

columns.

2 x 3

2 (rows), 3 (columns)

10. The array

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 0 & -5 \end{bmatrix}$$

is a .... x .... matrix, because it has .... rows and .... columns.

---

3 x 2

3, 2

Comment

Thus we have seen that the order of a matrix is given by stating first the number of rows and then the number of columns in the matrix. If the number of rows is the same as the number of columns, then the matrix is square. Our idea is to consider such an array of many numbers as a single object, an array, a matrix and to give the whole array a single name or symbol. By regarding a rectangular array of numbers as constituting a single object, a matrix, we will be able to handle large sets of numbers as single units, thereby simplifying the statement of complicated relationships. Thus, we might call our matrices of frames 8, 9, and 10 as A, B, C, and D.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 4 & 3 \\ -1 & 0 & 1 \end{bmatrix}, \quad \text{etc.}$$

11. If E is the 4 x 5 matrix

$$E = \begin{bmatrix} 1 & 2 & 1 & 9 & 10 \\ 0 & 7 & 8 & -2 & 12 \\ 5 & -3 & 4 & -5 & 15 \\ 43 & 17 & -10 & 11 & 22 \end{bmatrix}$$

the element in the second row and third column of E is 8 and it can be written conveniently as $[E]_{2,\ 3} = 8$. The element in the third row and fourth column is ...., and it can be written as $[E]_{\dots\dots,\ \dots\dots} = -5$.
(number

-5

3, 4

---

12. In the matrix of frame 11, the element in the fourth row and fourth column is ....

11

---

13. In the matrix of frame 11, $[E]_{3,\ 3} =$ ....

4

14. In the matrix of frame 11, $[E]_{4,\ \ldots\ldots\ldots}$ = 11

4

15. In the matrix of frame 11, the element of ith row and jth column can be written as $[E]_{\ldots\ldots,\ \ldots\ldots}$

i, j

16. In the matrix of frame 11, there are .... entries in one row.
(number)

5

17. In the matrix of frame 11, there are .... entries in one column.

4

18. In the 4 x 5 matrix of frame 11, there are .... entries altogether.

20

---

19. In a 3 x 3 matrix there are .... entries.

9

---

20. In a 3 x 4 matrix there are .... entries.

12

---

21. In an n x m matrix there are .... entries.

---

n m

---

22. In an n x n matrix there are .... entries.

---

n^2

23, Let I be the matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then

(a) $[I]_{4,4} =$ ....

1

(b) $[I]_{3,4} =$ ....

0

(c) $[I]_{4,3}$ = ...

0

(d) The elements of row three are ...., ..., ..., ..., ...

0, 0, 1, 0, 0

(e) The elements of column three are ...., ..., ..., ..., ...

0, 0, 1, 0, 0

(f) The element in both row three and column three is ....

1

(g) The element $[I]_{i,j}$ = ..., when i = j

1

(h) The element $[I]_{i,j}$ = ... when i ≠ j.

0

(i) There are .... entries in the matrix.

25

---

24. If we interchange the rows and columns of a matrix then the new matrix formed will be different from the original matrix and will be called its transpose.

If A is a matrix then A' will be its transpose.

For example, if

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 0 \end{bmatrix}, \quad \text{then } A' = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 0 \end{bmatrix}$$

The transpose of

$$B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 2 \\ 0 & 3 & -1 \end{bmatrix} \quad \text{is } B' = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 3 \\ 4 & 2 & -1 \end{bmatrix}$$

---

25. If a matrix is 3 x 4, then its transpose will be .... x .... matrix.

4 x 3

---

26. If in a 4 x 5 matrix there are 20 entries, then in its transpose there will be .... entries.

20

---

27. If there is an m x n matrix, then its transpose will be .... x .... matrix and there will be .... entries in the tranpose matrix.

n x m

n m or m n

Unit II: Equality of Matrices

1. If

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & 2 \\ -7 & 3 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & 2 \\ -7 & 3 & 0 \end{bmatrix}$$

then

(a) The size of A is .... to the size of B.
(word)

---

equal

(b) All the entries of A are the .... as the corresponding entries of B.
(word)

---

same

(c) The two matrices A and B are said to be equal if

(i) A and B are of the same size.

(ii) All the entries of A are the same as the corresponding entries of B.

Therefore matrices A and B are ....
(word)

---

equal

2. Now if

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & 2 \\ -7 & 3 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & 2 \\ -7 & 2 & 0 \end{bmatrix}$$

then

(a) The size of A is .... to the size of B.

----

equal

(b) The entry of row ...... and column .... of matrix A is different from the corresponding entry of B.

3, 2

(c) Therefore all the entries of A are .... (same/not the same) as the corresponding entries of B.

----

not the same

(d) Therefore the matrix A is .... to the matrix B.
(equal/not equal)

not equal

3. If

$$A = \begin{bmatrix} 4 & 5 & 1 \\ 2 & 5 & 2 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & 2 \\ 2 & 0 & -1 \end{bmatrix}$$

then

(a) The size of A is .... to the size of B.

not equal

(b) All the entries of A are .... as the corresponding entries of B.

not the same

(c) Therefore matrices A and B are ....

not equal

4. Let A be an $m \times m$ matrix, let B be an $n \times n$ matrix, and suppose $m \neq n$. Then A is .... to B.
(equal/not equal)

---

not equal

---

5. Let A be an $m \times m$ matrix, let B be an $n \times n$ matrix and suppose $m = n$, $[A]_{i,j} \neq [B]_{i,j}$ for all i and j between 1 and n. Then A is .... to B.
(equal/not equal)

---

not equal

---

6. Let A be an $m \times m$ matrix, let B be an $n \times n$ matrix, and suppose $m = n$, $[A]_{i,j} = [B]_{i,j}$ for all i and j between 1 and n. Then A is .... to B.
(equal/not equal)

---

equal

---

7. Using the foregoing definition of equality, we can express certain relationships more compactly, ----- For example, the equation.

$$\begin{bmatrix} x + 2y & a + 3b \\ x - y & a - b \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

can be written in place of the four equations

$$x + 2y = 4$$

$$x - y = 1$$

$$a + 3b = -3$$

$$a - b = 2$$

8. Express the following relationships in matrix form

$x + y = 3$

$y + z = 4$

$z + w = 5$

$w + x = 6$

$$\begin{bmatrix} & \\ & \\ & \\ & \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} x + y & z + w \\ y + z & w + x \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

9. Let A = B

$$\text{where } A = \begin{bmatrix} x + 2y & a - 3b \\ -x - y & 2a + 4b \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 35 \end{bmatrix}$$

then

(a) $x + 2y$ = ...

0

(b) $-x - y$ = ...

$\frac{1}{2}$

(c) $a - 3b =$ ....

1

(d) $2a + 4b =$ ....

35

---

10. Let

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad \text{then}$$

(a) $[B]_{i,j} = 0$ when i .... j
($<$ or $>$)

"$>$" (greater than)

(b) $[B]_{i,j} = 1$ when i .... j
($<$ or $\leq$)

"$\leq$" (less than or equal to)

11. The zero matrix is a matrix all of whose elements are the number 0. Thus, the 2 x 2 zero matrix is

$$0_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The symbol 0_n means the .... x .... zero matrix. Since one can always tell from context what size zero matrix is being considered, the subscript is usually excluded.

---

n x n

Unit III: Addition of Matrices

1. To make the study of matrices meaningful and useful we must now consider basic operations with matrices. In this unit we define and study sums of matrices. Products will be considered later.

Let the scores of Arithmetic and English of three students for the mid-term as well as for the final examination are represented by the following two matrices A and B.

$$A = \begin{array}{cc} \text{Arith} & \text{Eng} \\ \left[\begin{matrix} 6 \\ 8 \\ 7 \end{matrix}\right. & \left.\begin{matrix} 8 \\ 10 \\ 8 \end{matrix}\right] \end{array} \qquad B = \begin{array}{cc} \text{Arith} & \text{Eng} \\ \left[\begin{matrix} 9 \\ 9 \\ 8 \end{matrix}\right. & \left.\begin{matrix} 8 \\ 7 \\ 6 \end{matrix}\right] \end{array}$$

Matrix A represents the scores of mid-term and matrix B represents the scores of the final examination. Now to find out the sum of scores on both the examinations for each student, matrices A and B will be added therefore

$$A + B = \begin{bmatrix} 6 & 8 \\ 8 & 10 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 9 & 7 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 6+9 & 8+8 \\ 8+9 & 10+7 \\ 7+8 & 8+6 \end{bmatrix} = \begin{bmatrix} 15 & 16 \\ 17 & 17 \\ 15 & 14 \end{bmatrix}$$

The rule for the addition of matrices is: The matrices of the same size are added by adding each element of one matrix to the corresponding element of the other matrix. Thus,

$$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 6 & 7 \\ 2 & 5 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 8 & 9 \\ 11 & 13 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 16 \\ 12 & 15 \end{bmatrix}$ and C represents the sum of A and B, then:

(a)

$$C = \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$$

$$\begin{bmatrix} 13 & 25 \\ 23 & 28 \end{bmatrix}$$

(b) The size of the matrix A is ....

---

2 X 2 or 2

(c) The size of the matrix B is ....

---

2 X 2 or 2

(d) The size of the matrix C is ....

---

2 X 2 or 2

3. Let

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 3 & -2 \end{bmatrix}$$

and let C be the sum of A and B, so that

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \quad ; \quad \text{then}$$

(a) The value of c_{11} = ...

-2

(b) The value of c_{22} = ...

3

(c) The size of the matrices A and B is ....

----

2 X 3

(d) The size of the matrix C is ....

2 X 3

4. The sum A + B of two m x n matrices A and B is the m x n matrix C whose entry in the ith row and jth column is the sum of $[A]_{i,j}$ and $[B]_{i,j}$.

Therefore,

(a) $[A + B]_{i,j} = [A]_{i,j} + \ldots\ldots\ldots\ldots$

$[B]_{i,j}$

(b) $[A]_{i,j} + [B]_{i,j} = [A + B]_{i,j} = \ldots\ldots\ldots\ldots\ldots\ldots$ (entry of matrix C)

$[C]_{i,j}$

5. If

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

then

(a) c_{11} = ... + ...

- -

$a_{11} + b_{11}$

(b) c_{12} = ...

- -

$a_{12} + b_{12}$

(c) c_{21} = ...

- -

$a_{21} + b_{21}$

(d) c_{22} = ...

$a_{22} \div b_{22}$

(e) c_{31} = ...

$a_{31} \div b_{31}$

(f) c_{32} = ...

$a_{32} \div b_{32}$

6. Let

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \qquad \text{and } 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then

(a)

$$A \div 0 = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

(write four entries)

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

(b) Therefore $A + 0 =$ ....
(name of the matrix)

A

---

7. Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -3 & 4 \end{bmatrix} \text{ and } 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ then}$$

$$A + 0 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix} = \text{........}$$

(write entries) (name of the matrix)

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -3 & 4 \end{bmatrix}, \ A$$

---

8. If A is an m x n matrix and 0 which is a zero matrix is also of size m x n, then

$A + 0 =$ ....

A

Comment

We have seen that if a zero matrix is added to another matrix (of course, both of the same size), then the resultant matrix is equal to the non-zero matrix. Therefore in matrix algebra zero matrix in addition is identity matrix as in the algebra of real numbers 0 is the identity element for addition $(0 + a = a)$

9.

Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$; then

(a)

$$A + B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ 4 & 4 \end{bmatrix}$$

---

and

(b) $$B + A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ 4 & 4 \end{bmatrix}$$

(c) Therefore A + B = ... + ...
(name the matrices)

B + A

10. Now let $C = \begin{bmatrix} 1 & 3/2 & 1/3 \\ 0 & 1 & 0 \\ -1 & -1 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & -1/2 & 2/3 \\ 1 & -1 & 2 \\ 1 & 1 & -6 \end{bmatrix}$

Then

(a)

$$C + D = \begin{bmatrix} 1 & 3/2 & 1/3 \\ 0 & 1 & 0 \\ -1 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & 2/3 \\ 1 & -1 & 2 \\ 1 & 1 & -6 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

(b)

$$\text{and } D + C = \begin{bmatrix} 0 & -1/2 & 2/3 \\ 1 & -1 & 2 \\ 1 & 1 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 3/2 & 1/3 \\ 0 & 1 & 0 \\ -1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

(c) Therefore C + D = ... + ...

D + C

11. Let A and B be two matrices of size m x n so that

$A + B = C$

and $B + A = D$

then

(a) the entry in the ith row and jth column of (A + B) will be

$[A + B]_{i,j}$ = ... + ...

$[A]_{i,j} + [B]_{i,j}$

(b) and the entry in the ith row and jth column of (B + A) will be

$[B + A]_{i,j}$ = ... + ...

$[B]_{i,j} + [A]_{i,j}$

(c) The elements of a matrix are numbers, $[A]_{i,j}$ and $[B]_{i,j}$ are elements of ith row and jth column of matrices A and B, and therefore are ....

numbers.

(d) Numbers are commutative with respect to addition, as $2 + 3 = 3 + 2$ or more generally $a + b = b + a$. Therefore $[A]_{i,j}$ and $[B]_{i,j}$ are .... in addition.

commutative

(e) Therefore:

$[A]_{i,j} + [B]_{i,j}$ and $[B]_{i,j} + [A]_{i,j}$ are ....
(equal or unequal)

equal

(f) But

$$[A]_{i,j} + [B]_{i,j} = [C]_{i,j}$$

$$[B]_{i,j} + [A]_{i,j} = [D]_{i,j}$$

Therefore $[C]_{i,j}$ and $[D]_{i,j}$ are .... and therefore C = D.

---

equal

(g) When C and D are equal then (A + B) and (B + A) are ....

---

equal

---

12. We have seen that matrices A + B = B + A and therefore matrices of the same size are commutative in addition. It is also true in the algebra of real numbers where numbers are commutative in addition, e.g., a + b = ... + ...

---

b + a

---

13. Let $A = \begin{bmatrix} 1/2 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1/6 & 4 \\ 0 & -1/3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -1/3 \\ 1/4 & 1 \end{bmatrix}$

then

(a) $B + C = \begin{bmatrix} -1/6 & 4 \\ 0 & -1/3 \end{bmatrix} + \begin{bmatrix} 0 & -1/3 \\ 1/4 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$$\begin{bmatrix} -1/6 & 11/3 \\ 1/4 & 2/3 \end{bmatrix}$$

and

(b) $A + B + C = A + (B + C) = \begin{bmatrix} 1/2 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1/6 & 11/3 \\ 1/4 & 2/3 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$$\begin{bmatrix} 1/3 & 17/3 \\ 1/4 & -1/3 \end{bmatrix}$$

(c) Also

$A + B = \begin{bmatrix} 1/2 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1/6 & 4 \\ 0 & -1/3 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$$\begin{bmatrix} 1/3 & 6 \\ 0 & -4/3 \end{bmatrix}$$

(d) And A + B + C = (A + B) + C = $\begin{bmatrix} 1/3 & 6 \\ 0 & -4/3 \end{bmatrix} + \begin{bmatrix} 0 & -1/3 \\ 1/4 & 1 \end{bmatrix} = \begin{bmatrix} \text{....} & \text{....} \\ \text{....} & \text{....} \end{bmatrix}$

$$\begin{bmatrix} 1/3 & 17/3 \\ 1/4 & -1/3 \end{bmatrix}$$

(e) Therefore

A + (B + C) and (A + B) + C are ....

equal

14. Let A, B, and C be m x n matrices.

Then

(a) $[(A + B) + C]_{i,j} = [A + B]_{i,j} + [C]_{i,j}$ = ... + ... + ...

$[A]_{i,j} + [B]_{i,j} + [C]_{i,j}$

(b) And $[A + (B + C)]_{i,j} = [A]_{i,j} + [B + C]_{i,j}$

= .... + ... + ...

$[A]_{i,j} + [B]_{i,j} + [C]_{i,j}$

(c) Now every element of $(A + B) + C$ is equal to the corresponding element of $A + (B + C)$ and both sums are $m \times n$ matrices (are of the same size), so $(A + B) + C$ and $A + (B + C)$ are ....

---

equal

---

15. We have seen that in matrices $(A + B) + C = A + (B + C)$ and therefore matrices of the same size are associative in addition. It is also true in the algebra of real numbers where numbers are associative in addition, e.g. $(a + b) + c$ = ... + (...... + ...).

$a + (b + c)$

Unit IV: Subtraction of Matrices

1. Let A be an m x n matrix. Then the negative matrix of A is the matrix each of whose entries has the opposite sign of the corresponding entry of A. The negative matrix of A will be denoted by the symbol -A. That is, the negative matrix -A is defined by $-[A]_{i,j} = [-A]_{i,j}$.

If A and B are matrices of the same size, then the difference of A and B, denoted by A-B, is the sum of A and the negative of B. That is, A-B is defined as A + (-B).

If

$$A = \begin{bmatrix} -1/2 & 4 \\ 0 & 1 \end{bmatrix}$$

then

(a) $$-A = -\begin{bmatrix} -1/2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & -4 \\ 0 & -1 \end{bmatrix}$$

(b)

$$A + (-A) = \begin{bmatrix} -1/2 & 4 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1/2 & -4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix} = \text{.....}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(c)

$$-(-A) = -\left\{-\begin{bmatrix}-1/2 & 4\\ 0 & 1\end{bmatrix}\right\} = -\begin{bmatrix}1/2 & -4\\ 0 & -1\end{bmatrix} = \begin{bmatrix}..... & .....\\ ..... & .....\end{bmatrix} = ...$$

$$\begin{bmatrix}-1/2 & 4\\ 0 & 1\end{bmatrix} = A$$

---

2. If $0 = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

then

$$-0 = -\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}..... & .....\\ ..... & .....\end{bmatrix} = ...$$

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0$$

---

3. Let $A = \begin{bmatrix}-1/2 & 4\\ 0 & 1\end{bmatrix}$ and $B = \begin{bmatrix}2 & 2\\ 1 & 2\end{bmatrix}$

then

(a) $$-(A+B) = -\left\{\begin{bmatrix}-1/2 & 4\\ 0 & 1\end{bmatrix} + \begin{bmatrix}2 & 2\\ 1 & 2\end{bmatrix}\right\} = -\begin{bmatrix}..... & .....\\ ..... & .....\end{bmatrix}$$

$$= \begin{bmatrix}..... & .....\\ ..... & .....\end{bmatrix}$$

$$-\begin{bmatrix}3/2 & 6\\ 1 & 3\end{bmatrix} = \begin{bmatrix}-3/2 & -6\\ -1 & -3\end{bmatrix}$$

(b)

$$(-A) + (-B) = \left\{-\begin{bmatrix}-1/2 & 4\\ 0 & 1\end{bmatrix}\right\} + \left\{-\begin{bmatrix}2 & 2\\ 1 & 2\end{bmatrix}\right\}$$

$$= \begin{bmatrix}\cdots\cdot & \cdots\cdot\\ \cdots\cdot & \cdots\cdot\end{bmatrix} + \begin{bmatrix}\cdots\cdot & \cdots\cdot\\ \cdots\cdot & \cdots\cdot\end{bmatrix} = \begin{bmatrix}\cdots\cdot & \cdots\cdot\\ \cdots\cdot & \cdots\cdot\end{bmatrix}$$

- -

$$\begin{bmatrix}1/2 & -4\\ 0 & -1\end{bmatrix} + \begin{bmatrix}-2 & -2\\ -1 & -2\end{bmatrix} = \begin{bmatrix}-3/2 & -6\\ -1 & -3\end{bmatrix}$$

(c) From the results of (a) and (b) what relation can be established between $-(A + B)$ and $(-A) + (-B)$?

....

- -

$$-(A + B) = (-A) + (-B)$$

---

4. If A is an m x n matrix, then

$$\left[A + (-A)\right]_{i,j} = \left[\cdots\cdot\cdot\right]_{i,j} + \left[\cdots\cdot\right]_{i,j} = \left[\cdots\cdot\right]_{i,j} - \left[\cdots\cdot\right]_{i,j} = \cdots\cdot$$

- -

A, -A ; A, A; 0

---

5. In the above frames we have seen that

$$A + (-A) = 0$$

It is true for any size of A.

---

6. If

$$\begin{bmatrix} x + 3 & 2y - 8 \\ a + 1 & 4x + 6 \\ b - 3 & 3b \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ -3 & 2x \\ 2b + 4 & -21 \end{bmatrix}$$

then

(a) The value of x that satisfies the matrix relationship is ....

-3

(b) The value of y is ....

1

(c) The value of a is ....

-4

(d) The value of b is ....

-7

7. If

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -5 & 6 \\ 0 & 8 & -3 \\ 4 & 6 & 8 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} -3 & 4 & 8 \\ -2 & 6 & -1 \\ 0 & 2 & 3 \\ 4 & -1 & 8 \end{bmatrix}$$

then determine the entry in the sum A + B that is at the intersection of

(a) the 3rd row and 2nd column. ....

10

(b) the 1st row and 3rd column. ....

9

(c) the 4th row and 1st column. ....

8

8. Compute

$$\begin{bmatrix} 1/2 & 1/3 \\ 1/4 & 1/5 \end{bmatrix} + \begin{bmatrix} 1/6 & 1/7 \\ 1/8 & 1/9 \end{bmatrix} = \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$$

$$\begin{bmatrix} 2/3 & 10/21 \\ 3/8 & 14/45 \end{bmatrix}$$

9. Does the sum

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

make sense? ....

No

10. Does the sum

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

make sense? ....

Yes

11. Compute

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(Comment)

Insofar as only addition and subtraction are involved, the algebra of matrices is exactly like the ordinary algebra of numbers.

Suppose A and B are known matrices of the same size. Now consider the matrix equation $X + A = B$, where X is of the same size as A and B but is unknown. If this equation involved ordinary numbers rather than matrices we would simply add -A to both sides of the equation; i.e.,

$$X + A + (-A) = B + (-A)$$

$$\text{or } X = B - A$$

It happens that we can do the very same sort of thing with matrices.

12. If C and D are known matrices of the same size and $Y + C = D$, where Y is of the same size as C and D but unknown,

then Y = ... - ....

D - C

---

13. If

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \; ; \; D = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \qquad \text{and } Y = D - C$$

then $Y = \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ -1 & -3 \end{bmatrix}$$

---

14. Solve the equation

$$X - \begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}$$

for the matrix X.

$$X = \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$$

$$X = \begin{bmatrix} 4 & 1 \\ 5 & 10 \end{bmatrix}$$

---

15. Solve the equation

$$X - \begin{bmatrix} 3 & 0 & 4 \\ 1 & 2 & 0 \\ 3 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 1 & 9 \\ 8 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 3 \\ 0 & 2 & 1 \\ 9 & 3 & 7 \end{bmatrix}$$

for the matrix X

$$X = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 2 & 4 \\ 8 & 1 & 8 \\ 2 & 5 & 4 \end{bmatrix}$$

Unit V: Numerical Multiples of Matrices

1. Once we know how to add numbers, it is customary to define $2x$ as the sum $x + x$, $3x$ as the sum $2x + x$, etc. Fractional parts of x are defined by requiring that $(1/2)x + (1/2)x = x$, $(1/3)x + (1/3)x + (1/3)x = x$, etc. All of this can readily be done with matrices. If we add two equal matrices, the sum is clearly a matrix in which each entry is exactly twice the corresponding entry in the two given matrices. Thus

$$\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(3) \\ 2(-1) & 2(0) \end{bmatrix}$$

The sum $3A = 2A + A = A + A + A$ is equally clearly the matrix each of whose entries is exactly three times the corresponding entry in A. The equation $(1/2)A + (1/2)A = A$ defining the matrix $(1/2)A$ is clearly satisfied by the matrix each of whose entries is exactly one-half the corresponding entry of A.

2. Compute

$$3\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 6 & 9 \\ -3 & 0 \end{bmatrix}$$

3. Compute

$$1/2\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 \\ -1/2 & 0 \end{bmatrix}$$

4. Let A be an n by n matrix and x any real number. Then the product of A by the number x is the matrix whose entry in the ith row and the jth column is x times the corresponding entry $\left[A \right]_{i,j}$ in the matrix A. That is, the matrix xA is defined by the formula

$$\left[xA \right]_{i,j} = x \left[A \right]_{i,j}$$

Notice that the product of a matrix by a number is another matrix.

5. Let x = 5, y = 2 and $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

then

(a) $x(yA) = 5 \left\{ 2 \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \right\}$

$$= 5 \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$5 \begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 30 \\ 20 & 10 \end{bmatrix}$$

(b) and

$$(xy)A = (5 \times 2) \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = 10 \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 10 & 30 \\ 20 & 10 \end{bmatrix}$$

(c) From the results of Frames 5(a) and 5(b), x(yA) and (xy)A are ....

---

equal

---

6. **If A be an m x n matrix the relation x(yA) = (xy)A will still be true as is clear from the following**

$$\left[x(yA)\right]_{i,j} = x\left[yA\right]_{i,j} = x\left\{y\left[A\right]_{i,j}\right\}$$
$$= xy\left[A\right]_{i,j} = \left[(xy)A\right]_{i,j}$$

---

7. Again let x = 5, y = 2 and $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

then

(a)

$$(x + y)A = (5 + 2)\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = 7\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot \end{bmatrix}$$

$$\begin{bmatrix} 7 & 21 \\ 14 & 7 \end{bmatrix}$$

(b) And

$$xA + yA = 5\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix} + \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix} = \begin{bmatrix} 7 & 21 \\ 14 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 15 \\ 10 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix}$$

(c) From the results of Frames 7(a) and 7(b) we have

$$(x + y)A = (xA) + (yA)$$

and it is true also when A is an m x n matrix.

8. If

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

then

(a) $(-1)A = (-1)\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$

$$\begin{bmatrix} -1 & -3 \\ -2 & -1 \end{bmatrix}$$

(b) and

$$-A = -\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} -1 & -3 \\ -2 & -1 \end{bmatrix}$$

(c) From 8(a) and 8(b) we have

$$(-1)A = -A$$

and it is true also when A is an m x n matrix.

9. Now let $x = 5$, $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix}$

then

(a) $$x(A+B) = 5\left\{\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix}\right\}$$

$$= 5\begin{bmatrix} 3 & 7 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 15 & 35 \\ 5 & 5 \end{bmatrix}$$

and
(b)

$$xA + xB = 5\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + 5\begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} + \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

$$\begin{bmatrix} 5 & 15 \\ 10 & 5 \end{bmatrix} + \begin{bmatrix} 10 & 20 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 15 & 35 \\ 5 & 5 \end{bmatrix}$$

(c) Therefore from 9(a) and 9(b) we have

$$x(A+B) = xA + xB$$

and it is also true when A and B are m x n matrices as may be seen from the following:

$$\begin{aligned} \left[x(A + B)\right]_{i,j} &= x\left[A + B\right]_{i,j} = x\left\{\left[A\right]_{i,j} + \left[B\right]_{i,j}\right\} \\ &= x\left[A\right]_{i,j} + x\left[B\right]_{i,j} \\ &= \left[xA\right]_{i,j} + \left[xB\right]_{i,j} \\ &= \left[xA + xB\right]_{i,j} \end{aligned}$$

---

10. Also if x = 5, then

$$x \cdot 0_2 = 5\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} = 0_2$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2

11. It is true in general that any numerical multiple of the zero matrix is the zero matrix, whatever the size of the zero matrix may be. It can be proved easily by using our notation for the entries in a matrix as follows:

$$\left[x \cdot 0_n\right]_{i,j} = x \left[0_n\right]_{i,j} = x \cdot 0 = 0$$

Therefore $x \cdot 0_n = 0_n$

---

12. When

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

then

$$0 \cdot A = 0 \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = 0_2$$

- -

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

---

13. The product of any matrix by the number zero is the zero matrix, whatever the size of the matrix may be. It is seen from the following:

Using our notation for the entries in a matrix, we may simply write:

$$\left[0 \cdot A\right]_{i,j} = 0 \cdot \left[A\right]_{i,j} = 0$$

Thus $0 \cdot A = 0_n$

---

14. Let

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 5 \\ 6 & 9 & -1 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 5 & -1 & 0 \\ 7 & 8 & -1 \end{bmatrix}$$

then

(a)

$$3A = \begin{bmatrix} \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \end{bmatrix}; \quad 4B = \begin{bmatrix} \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \end{bmatrix};$$

$$\text{and } 2C = \begin{bmatrix} \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \end{bmatrix}$$

$$3A = \begin{bmatrix} 6 & 3 & -9 \\ 3 & 0 & 12 \end{bmatrix}; \quad 4B = \begin{bmatrix} 12 & 0 & 20 \\ 24 & 36 & -4 \end{bmatrix}; \quad 2C = \begin{bmatrix} 10 & -2 & 0 \\ 14 & 16 & -2 \end{bmatrix}$$

(b)

$$4B + 2C = \begin{bmatrix} \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \end{bmatrix}$$

$$\begin{bmatrix} 22 & -2 & 20 \\ 38 & 52 & -6 \end{bmatrix}$$

(c)

$$3A - 4B - 2C = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} -16 & 5 & -29 \\ -35 & -52 & 18 \end{bmatrix}$$

---

15. Solve the equation

$$X + \begin{bmatrix} 3 & 4 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix}$$

for the matrix X.

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

---

16. Solve the equation

$$Z - \begin{bmatrix} 1/2 & 1/4 \\ 1/6 & 1/8 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/4 \\ -1/6 & -1/8 \end{bmatrix} \qquad Z = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix} = \text{.....}$$

for the matrix Z

$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Unit VI: Multiplication of Matrices

So far we have defined and studied:

(1) Addition of matrices

(2) Subtraction of matrices

(3) Multiplication of a matrix by a number

Now we shall define and study the product of two matrices.

The definition of multiplication of two matrices will be clear from the following example:

1. Suppose in a University three departments--Education, History, and English--require teaching faculty in three categories--Professors, Associate Professors, and Assistant Professors. The following is the number of such personnel required in each department:

Departments	Professors	Associate Professors	Assistant Professors
Education	10	12	15
History	5	8	10
English	8	15	20

We can, if we like, arrange this table in the form of a matrix:

$$\begin{array}{ll|ccc|}
 & & \multicolumn{3}{c}{\text{Personnel}} \\
 & & \text{Professors} & \text{Associate Professors} & \text{Assistant Professors} \\
\hline
\text{Departments} & \text{Education} & 10 & 12 & 15 \\
 & \text{History} & 5 & 8 & 10 \\
 & \text{English} & 8 & 15 & 20 \\
\hline
\end{array} = A$$

Suppose now the College has to pay the following money (in thousand dollars for each personnel:

	Regular Salary	Summer Salary	Transportation
Professor	20	4	3
Associate Professor	15	3	2
Assistant Professor	12	2	1

Again, we can arrange our information in the form of a matrix:

$$\begin{array}{rl} & \overbrace{\begin{array}{ccc}\text{Regular Salary} & \text{Summer Salary} & \text{Transportation}\end{array}}^{\text{Expenditure}} \\ \text{Personnel}\left\{\begin{array}{l}\text{Professor} \\ \text{Assoc. Professor} \\ \text{Assist. Professor}\end{array}\right. & \left[\begin{array}{ccc} 20 & 4 & 3 \\ 15 & 3 & 2 \\ 12 & 2 & 1 \end{array}\right] = B \end{array}$$

It is clear that by combining the above two sets of information, we can easily compute the expenditure of each department for the three categories of expenditure. That is, by correctly combining the entries in the two matrices, we can figure out another matrix which will have only departments and expenditures.

(a) For instance, to compute the expenditure of regular salaries for the Department of Education, we figure as follows:

Department of Education will spend for the regular salaries for: 10 Professors at the rate of 20 thousand dollars for one + 12 Associate Professors at the rate of 15 thousand dollars for one + 15 Assistant Professors at the rate of 12 thousand dollars for one

= 10 x 20 + 12 x 15 + 15 x 12 = 560 thousand dollars

We multiply each entry in the Education row of the matrix A (reading from left to right) by the corresponding element in the Regular Salary column of the matrix B (reading from top to bottom) and then added all the resulting products. This procedure is perfectly general.

(b) Thus, to find the expenditure of summer salaries for the Department of Education, we multiply each element in the Education row of the matrix A by the corresponding element in the Summer Salary column of the matrix B and then add, getting a total of

10 x 4 + 12 x 3 + 15 x 2 = 106 thousand dollars

Look at the example above and answer the following:

(c) To find the expenditure of the Department of Education for paying transportation, multiply each element in the Education row of the matrix A by the corresponding element in the Transportation column of the matrix B and then add, getting a total of

.....x.....+.....x.....+.....x.....=69 thousand dollars

---

10 x 3 + 12 x 2 + 15 x 1

(d) To find the expenditure of the History Department for regular salaries, we multiply each element in the ------- row of the matrix A by the corresponding element in the --------------------column of the matrix B and then add, getting a total of

.....x.....+.....x.....+.....x.....=340 thousand dollars

History; regular salary

5 x 20 + 8 x 15 + 10 x 12

(e) Find the expenditure of the History Department for summer salaries.

.....x.....+.....x.....+.....x.....=.....thousand dollars

5 x 4 + 8 x 3 + 10 x 2 = 64

(f) Find the expenditure of the History Department for transportation.

.....x.....+.....x.....+.....x.....=.....thousand dollars

5 x 3 + 8 x 2 + 10 x 1 = 41

(g) The values which we have in (a) to (f) are some of the elements of the matrix which is the product of matrices A and B. We can write these matrices as follows:

$$\begin{bmatrix}10 & 12 & 15\\ 5 & 8 & 10\\ 8 & 15 & 20\end{bmatrix} \times \begin{bmatrix}20 & 4 & 3\\ 15 & 3 & 2\\ 12 & 2 & 1\end{bmatrix} = \left[\begin{array}{r} 10 \times 20 + 12 \times 15 + 15 \times 12 = 560 = a\\ 5 \times 20 + 8 \times 15 + 10 \times 12 = 340 = d\\ h\end{array}\right.$$

$$\left.\begin{array}{rr} 10 \times 4 + 12 \times 3 + 15 \times 2 = 106 = b & 10 \times 3 + 12 \times 2 + 15 \times 1 = 69 = c\\ 5 \times 4 + 8 \times 3 + 10 \times 2 = 64 = e & 5 \times 3 + 8 \times 2 + 10 \times 1 = 41 = f\\ i & j\end{array}\right]$$

2. In the above frame the value of h can be obtained by multiplying each element

in the........row of the first matrix by the corresponding element in the
(which row)
........column of the second matrix and then by adding.
(which column)

third
first

3. In the matrix above [frame 1(g)],

(a) the value of i is

.....x.....+.....x.....+.....x.....=117

8 x 4 + 15 x 3 + 20 x 2

(b) the value of j is

.....x.....+.....x.....+.....x.....=.....

8 x 3 + 15 x 2 + 20 x 1 = 74

C O M M E N T

The most concise description of the process of multiplication is: "Multiply row by column." Very simply the rule is to multiply entries of a row by corresponding entries of a column and then add the products. Thus, given two matrices A and B, to find the entry in the ith row and jth column of the product matrix AB, multiply each entry in the ith row of the left-hand factor A by the corresponding entry in the jth column of the right-hand factor B, and then add all the resulting terms. If A and B are 4 x 4 matrices, then the entry in the ith row and jth column of the combination matrix AB will be

$$[A]_{i,1}[B]_{1,j} + [A]_{i,2}[B]_{2,j} + [A]_{i,3}[B]_{3,j} + [A]_{i,4}[B]_{4,j}$$

4.

$$\begin{array}{cccc} A & B & & AB \\ \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} & \begin{bmatrix}5 & 0\\ 6 & 7\end{bmatrix} & = & \begin{bmatrix}(1 \times 5) + (2 \times 6) & (1 \times 0 + 2 \times 7)\\ (3 \times 5) + (4 \times 6) & \underset{\text{write numbers}}{(\ldots\ldots\times\ldots\ldots) + (\ldots\ldots\times\ldots\ldots)}\end{bmatrix}\end{array}$$

3 x 0 + 4 x 7

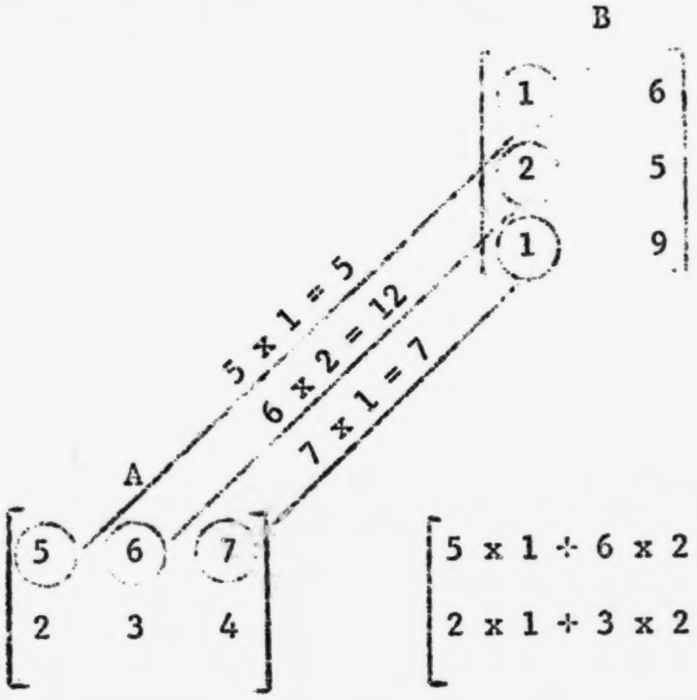
5.

$$\begin{array}{cccc} A & B & & AB \\ \begin{bmatrix}1 & 0 & -1\\ 2 & 3 & 4\\ -4 & 5 & -2\end{bmatrix} & \begin{bmatrix}4 & 3 & 2\\ 7 & 0 & -6\\ -3 & -5 & 6\end{bmatrix} & = & \begin{bmatrix}(4 + 0 + 3) & (3 + 0 + 5) & \underset{\text{(write numbers)}}{(\ldots\ldots+\ldots\ldots-\ldots\ldots)}\\ (8 + 21 - 12) & (6 + 0 - 20) & (4 - 18 + 24)\\ (-16 + 35 + 6) & \underset{\text{(write numbers)}}{(\ldots+\ldots+\ldots)} & (-8 - 30 - 12)\end{bmatrix}\end{array}$$

2 + 0 - 6

-12 + 0 + 10

6. Let us try the following schematic device to explain multiplication:



AB

$$\begin{bmatrix} 5 \times 1 + 6 \times 2 + 7 \times 1 & 5 \times 6 + 6 \times 5 + 7 \times 9 \\ 2 \times 1 + 3 \times 2 + 4 \times 1 & \ldots\times\ldots+\ldots\times\ldots+\ldots\times\ldots \\ & \text{(write numbers)} \end{bmatrix}$$

---

2 x 6 + 3 x 5 + 4 x 9

We see that the matrix A is of size 2 x 3 while the size of B is 3 x 2. Though the matrices A and B are not equal, yet there is an entry in each row of the matrix A to match with each entry in a column of the matrix B, and conversely. It follows that the product is not defined unless the number of columns in the A matrix is equal to the number of rows in the B matrix. When the number of columns in the left-hand factor equals the number of rows in the right-hand factor, the matrices are *conformable for multiplication*.

7. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 1 \end{bmatrix}$$

then

(a) The number of columns in the matrix A is ...., and the number of rows in the matrix B is ....., therefore, matrices A and B are

_________________ for multiplication.
(conformable or unconformable)

3, 3, conformable

(b)

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 17 & 5 \\ 38 & 11 \\ 59 & 17 \end{bmatrix}$$

---

8. Perform the following matrix multiplication:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

$$\begin{bmatrix} 17 & 5 & 14 \\ 38 & 11 & 32 \\ 59 & 17 & 50 \end{bmatrix}$$

9. Perform the following matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

$$\begin{bmatrix} 9 & 2 & 5 \\ 7 & 6 & 10 \\ 7 & -2 & -1 \end{bmatrix}$$

10. Perform the following matrix multiplication:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

COMMENT

Let A and B be two matrices of order m x p and p x n, respectively. The product AB is the matrix of order m x n, of which the entry in the ith row and the jth column is the sum of the products formed by multiplying entries of the ith row of A by corresponding entries of the jth column of B. The definition of the product of two matrices can be expressed in terms of the "$\sum$ notation" for sums. Recall that, in the "$\sum$ notation," we write the sum

$$S = x_1 + x_2 + \ldots . + x_p$$

of p numbers as

$$S = \sum_{j=1}^{p} x_j$$

In this notation, the product AB of two matrices A and B of order m x p and p x n, respectively is the matrix whose entry in the ith row and jth column is

$[AB]_{i,j} = \sum_{k=1}^{p} [A]_{i,k} \ [B]_{k,j}$; which by expansion becomes

$$= [A]_{i,1} \ [B]_{1,j} + [A]_{i,2} \ [B]_{2,j} + \ldots . + [A]_{i,p} \ [B]_{p,j}$$

Unit VII: Properties of Matrix Multiplication

1. Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then,

(a) $$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

---

and

(b) $$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

2\.

In the previous frame we saw that $AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, therefore AB and BA are ....
(equal/not equal)

(not equal)

---

3. Let

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & 2 \\ -1 & 10 \end{bmatrix}$$

then

(a) $$AB = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 10 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -3 & 18 \end{bmatrix}$$

---

and

(b) $$BA = \begin{bmatrix} 1 & 2 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ -11 & 20 \end{bmatrix}$$

4.

We saw in the previous frame that $AB = \begin{bmatrix} 1 & 2 \\ -3 & 18 \end{bmatrix}$ and $BA = \begin{bmatrix} -1 & 4 \\ -11 & 20 \end{bmatrix}$,

therefore AB and BA are....................
(equal/not equal)

---

not equal

5. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

then

(a)

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \end{bmatrix}$$

$$\begin{bmatrix} 9 & 2 & 5 \\ 7 & 6 & 10 \\ 7 & -2 & -1 \end{bmatrix}$$

---

and

(b) $$BA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$$

$$\begin{bmatrix} 4 & 10 \\ 3 & 10 \end{bmatrix}$$

6. Again we saw that

$$AB = \begin{bmatrix} 9 & 2 & 5 \\ 7 & 6 & 10 \\ 7 & -2 & -1 \end{bmatrix}, \quad \text{and } BA = \begin{bmatrix} 4 & 10 \\ 3 & 10 \end{bmatrix}, \text{ therefore AB and BA}$$

are ....matrices
(equal/different)

---

different (or not equal)

COMMENT

We have seen that if there are two matrices A and B which are conformable for multiplication, then their product AB is not equal to BA. Thus we have a first difference between matrix algebra and ordinary algebra, and a very significant difference it is indeed. When we multiply real numbers, we can rearrange factors since the commutative law holds, e.g., for all values of x and y, $xy = yx$. When multiplying matrices, we have no such law. We must, consequently distinguish between the result of multiplying B on the right by A to get BA and the result of multiplying B on the left by A to get AB. In the algebra of numbers, these two operations of "right multiplication" and "left multiplication" are the same; in matrix algebra, they are entirely different. Another way of putting this is: in general *multiplication of two matrices is not commutative*.

---

7. Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Here $A \neq 0$ and $B \neq 0$.

But
$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix} = \text{.....}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

---

8. Let

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}$$

Here again $A \neq 0$ and $B \neq 0$.

But

$$AB = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix} = \text{.....}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

9. Again, let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

C O M M E N T

We have seen that the product of two matrices can be zero without either of the two matrices being zero. This is a second major difference between ordinary algebra and matrix algebra. In ordinary algebra xy = 0 only if either x or y is zero, but it is not true in matrix algebra because AB can be zero even if neither A nor B is zero.

10. Now let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

then

(a)

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix}$$

and

(b)

$$AC = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix}$$

---

11. In the previous frame we found that

$$AB = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix}, \quad \text{and } AC = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix}$$

therefore AB and AC are ....
(equal/not equal)

---

equal

Here we see that $AB = AC$, while $A \neq 0$, and $B \neq C$, while in ordinary algebra if $ab = ac$ then either $a = 0$ or $b = c$ (cancellation law for multiplication). Therefore, the breakdown for matrix algebra of the law that $xy = yx$ and of the law that $xy = 0$ only if either x or y is zero causes additional differences. The cancellation law for multiplication for ordinary algebra can be proved as follows:

(a) $ab = ac$

(b) $ab - ac = 0$,

(c) $a(b-c) = 0$,

(d) $b-c = 0$,

(e) $b = c$.

For matrices, the above step from (c) to (d) fails and proof is not valid for matrices.

COMMENT

Let us consider another difference. We know that a real number 'a' can have at most two square roots; that is, there are at most two roots of the equation $xx = a$.

Proof. Again, we give the simple steps of the proof:

(a) Suppose that $yy = a$; then

(b) $xx = yy$,

(c) $xx-yy = 0$,

(d) $(x-y)(x+y) = xx + (-yx + xy) - yy$,

(e) $yx = xy$.

(f) From (d) and (e), $(x-y)(x+y) = xx-yy$.

(g) From (c) and (f), $(x-y)(x+y) = 0$.

(h) Therefore, either $x-y = 0$ or $x + y = 0$

(i) Therefore, either $x = y$ or $x = -y$.

For matrices in general, statement (e) is false, and therefore the steps to (f) and (g) are invalid. Even if (g) were valid, the step from (g) to (h) fails. So the proof is completely wrong if we try to apply it to matrices. In fact, it is false that a matrix can have at most two square roots.

---

12. Multiply the following:

$$\begin{bmatrix} 0 & x \\ 1/x & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 1/x & 0 \end{bmatrix} = \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 0 & x \\ 1/x & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & x \\ 1/x & 0 \end{bmatrix}$ are the square roots of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If we give different values (except 0) to x, we can get different square roots of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus the very simple 2 x 2 matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has infinitely many distinct square roots.

---

13. Let A and B be two matrices of equal size, then

$$(A + B)\ (A + B) = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2,$$

because in general AB and BA are ....
(equal/not equal)

---

not equal

---

14. Let A and B be two matrices of equal size, then

$(A + B)\ (A - B) = A^2 - AB + BA - B^2 \neq A^2 - B^2$, because in general AB and BA are ....
(equal/not equal)

---

not equal

---

15. Let

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \text{ ; then}$$

(a) the sum of A and B is

$$A + B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix}$$

(b)

$$A-B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

(c)

$$AB = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix}$$

(d)

$$A^2 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}$$

(e) $$B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

(f)

$$(A + B)\ (A + B) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 3 & -6 \\ 6 & 15 \end{bmatrix}$$

(g)

$$A^2 + 2\ AB + B^2 = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} + 2 \begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 2 & -7 \\ 7 & 16 \end{bmatrix}$$

(h) Now

$$(A + B)\,(A + B) = \begin{bmatrix} 3 & -6 \\ 6 & 15 \end{bmatrix}, \quad \text{and} \quad A^2 + 2AB + B^2 = \begin{bmatrix} 2 & -7 \\ 7 & 16 \end{bmatrix}$$

therefore $(A + B)\,(A + B)$ is ....to $A^2 + 2AB + B^2$
(equal/not equal)

not equal

(i)

$$(A + B)\,(A - B) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -4 & -1 \end{bmatrix}$$

(j)

$$A^2 - B^2 = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

(k) Now

$(A + B)\,(A - B) = \begin{bmatrix} 1 & -2 \\ -4 & -1 \end{bmatrix}$, and $A^2 - B^2 = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$ then $(A + B)\,(A - B)$

is .... to $A^2 - B^2$.
(equal/not equal)

not equal

16. Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

then

$$A^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix} = \text{.....}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

17. Let

$$A = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} \quad (n = \text{any real number})$$

then

$$A^2 = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix} = \text{.....}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

18. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

then

(a) $$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}$$

(b)

$$BA = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

(c)

$$(AB)A = \begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 11 & 14 \end{bmatrix}$$

(d)

$$(BA)A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 7 & 10 \\ 8 & 12 \end{bmatrix}$$

19. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \\ 5 & 1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then

(a) $AI = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} = A$

$$\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \\ 5 & 1 & 2 \end{vmatrix}$$

---

and

(b) $IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \\ 5 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} =$ ....
(name of the matrix)

$$\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \\ 5 & 1 & 2 \end{vmatrix} = A$$

Therefore

(c) $IA = A = AI$

20. (a) Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$AI = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \ldots\ldots\ldots$$

(name of the matrix)

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = A$$

(b) Now let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

then

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \ldots\ldots\ldots$$

(name of the matrix)

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = A$$

(c) Therefore

$$AI = A = IA$$

COMMENT

We have seen above that the matrix I when multiplied to the matrix A (either on the left or on the right) just gives A back again. The matrix I is called the unit matrix, or the identity matrix for multiplication. The unit matrix for the multiplication of matrices plays the same role as the number 1 does in the multiplication of real numbers. (For all real numbers a, $1a = a = a1$.) The matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is called the unit matrix of size 2. The matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is called the unit matrix of size 3, and so forth.

In general, the unit matrix of size n, which we may denote by the symbol I_n is the matrix whose entries are

$$[I_n]_{i,j} = 0 \text{ if } i \neq j$$

$$[I_n]_{i,j} = 1 \text{ if } i = j \text{, that is } [I]_{j,j} = 1 \text{ for every } j.$$

Now let A be an n x n matrix. Then by definition of multiplication, the entry in the ith row and jth column of the product AI is

$$[A]_{i,1}[I]_{1,j} + [A]_{i,2}[I]_{2,j} + \dots\dots\dots\dots + [A]_{i,n}\ [I]_{n,j}$$

Since $[I]_{k,j} = 0$ whenever k is different from j, every term but one in this last expression is equal to zero and drops out, and we are left with just one term: $[A]_{i,j}\ [I]_{j,j}$, since $[I]_{j,j} = 1$, $[A]_{i,j}\ [I]_{j,j} = [A]_{i,j}$. Thus the entry in the ith row and jth column of the product AI is simply $[A]_{i,j}$. Thus AI and A have exactly the same entries, and therefore $AI = A$.

Similarly

$$\begin{aligned}[IA]_{i,j} &= [I]_{i,1}[A]_{1,j} + [I]_{i,2}[A]_{2,j} + \dots\dots\dots\dots + [I]_{i,n}[A]_{n,j}\\ &= [I]_{i,i}[A]_{i,j}; \quad \text{Since } [I]_{i,k} = 0 \text{ when } k \neq i\\ &= [A]_{i,j}; \quad \text{Since } [I]_{i,i} = 1\end{aligned}$$

Thus IA = A

and therefore AI = A = IA.

---

21. Compute

$$\begin{bmatrix} 0 & 0 & n \\ 0 & 1 & 0 \\ 1/n & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & n \\ 0 & 1 & 0 \\ 1/n & 0 & 0 \end{bmatrix} = \begin{bmatrix} \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \end{bmatrix} = \dots\dots\dots\dots$$

(name of the matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

---

22. In the previous frame we saw that

$$\begin{bmatrix} 0 & 0 & n \\ 0 & 1 & 0 \\ 1/n & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & n \\ 0 & 1 & 0 \\ 1/n & 0 & 0 \end{bmatrix} = I$$

therefore $\begin{bmatrix} 0 & 0 & n \\ 0 & 1 & 0 \\ 1/n & 0 & 0 \end{bmatrix}$ is the .... of I

square root

Notice that giving infinitely many values (except 0) to n, I can have infinitely many square roots.

23. Let

$$A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then

(a)

$$AI = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} = \text{.........}$$

(name of the matrix)

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} = A$$

(b)

$$(AI)I = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} = \text{.......}$$

(name of the matrix)

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} = A$$

(c)

$$I \cdot I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} = \text{.......}$$

(name of the matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

(d)

$$A(I \cdot I) = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix} = ...$$

(name of the matrix)

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} = A$$

(e)

$$((AI)I)I = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix} = ...$$

(name of the matrix)

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} = A$$

Unit VIII: The Laws of Matrix Multiplication

We saw in the previous unit that two basic laws which govern multiplication in the algebra of ordinary numbers break down when it comes to matrices. That is, we are faced with the

Breakdown of the commutative law: The product AB of two matrices may be entirely different from the product BA of the same two matrices.

Breakdown of the law of cancellation: The product AB of two matrices may be zero even if both factors A and B are other than zero.

Aside from these two laws, most of the other basic laws of ordinary algebra remain valid for matrices. These laws will be stated in this unit.

---

1. Let

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 1/2 & 3 \\ -3 & 1 \end{bmatrix}$$

then

(a)

$$AB = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

- -

$$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}$$

(b)

$$(AB)C = \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

- -

$$\begin{bmatrix} -14 & -8 \\ 0 & 0 \end{bmatrix}$$

(c)

$$BC = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} -7 & -4 \\ 0 & 0 \end{bmatrix}$$

(d) $$A(BC) = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -7 & -4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} -14 & -8 \\ 0 & 0 \end{bmatrix}$$

---

2. In the previous frame we saw that

$$(AB)C = \begin{bmatrix} -14 & -8 \\ 0 & 0 \end{bmatrix}, \text{ and } A(BC) = \begin{bmatrix} -14 & -8 \\ 0 & 0 \end{bmatrix}$$

therefore (AB)C is ....to A(BC).

----

equal

AB(C) = A(BC) is also true in general. It is similar to the associative law of multiplication of ordinary algebra where (ab)c = a(bc).

3. Again let

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1/2 & 3 \\ -3 & 1 \end{bmatrix}$$

then

(a)

$$B + C = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1/2 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} -3/2 & 5 \\ -3 & 1 \end{bmatrix}$$

(b)

$$A(B + C) = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3/2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} -15 & 14 \\ -3 & 1 \end{bmatrix}$$

(c)

$$AB = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}$$

(d)

$$AC = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$$

$$\begin{bmatrix} -11 & 10 \\ -3 & 1 \end{bmatrix}$$

(e)

$$AB + AC = \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -11 & 10 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$$

$$\begin{bmatrix} -15 & 14 \\ -3 & 1 \end{bmatrix}$$

(f) Now

$$A(B + C) = \begin{bmatrix} -15 & 14 \\ -3 & 1 \end{bmatrix}, \quad \text{and } AB + AC = \begin{bmatrix} -15 & 14 \\ -3 & 1 \end{bmatrix}$$

therefore $A(B + C)$ is .... to $AB + AC$.

equal

Also

(g)

$$(B + C)A = \begin{bmatrix} -3/2 & 5 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 \\ -6 & -11 \end{bmatrix}$$

(h)

$$BA = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} -4 & -6 \\ 0 & 0 \end{bmatrix}$$

(i)

$$CA = \begin{bmatrix} 1/2 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 \\ -6 & -11 \end{bmatrix}$$

(j)

$$BA + CA = \begin{bmatrix} -4 & -6 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ -6 & -11 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 \\ -6 & -11 \end{bmatrix}$$

(k) Now

$$(B + C)A = \begin{bmatrix} -3 & -1 \\ -6 & -11 \end{bmatrix}, \text{ and } BA + CA = \begin{bmatrix} -3 & -1 \\ -6 & -11 \end{bmatrix}$$

therefore $(B + C)A$ is .... to $BA + CA$.

equal

4. In the previous frame we figured out that

$$A(B + C) = \begin{bmatrix} -15 & 14 \\ -3 & 1 \end{bmatrix}, \text{ and } (B + C)A = \begin{bmatrix} -3 & -1 \\ -6 & -11 \end{bmatrix}$$

therefore

$A(B + C)$ and $(B + C)A$ are ....

not equal

5. Also we figured out previously that

$$AB + AC = \begin{bmatrix} -15 & 14 \\ -3 & 1 \end{bmatrix}, \text{ and } BA + CA = \begin{bmatrix} -3 & -1 \\ -6 & -11 \end{bmatrix}$$

therefore

$(AB + AC)$ and $BA + CA$ are ....

not equal

COMMENT

If A, B and C are matrices of the same size then $A(B + C) = AB + AC$ which is called left distributive law of multiplication with respect to addition and is similar to the ordinary algebra distributive law ($a(b + c) = ab + ac$).

And

$(B + C)A = BA + CA$ which is called right distributive law of multiplication. Since multiplication of matrices is not commutative, we cannot conclude that the left distributive multiplication is equal to the right distributive multiplication, i.e., $A(B + C) \neq (B + C)A$, though it is true in ordinary algebra where multiplication is commutative and therefore

$$a(b + c) = (b + c)a.$$

6. Let A, B, and C be n x n matrices. Then

$$\begin{aligned}
\left[(A(B + C)\right]_{i,j} &= [A]_{i,1}[B + C]_{1,j} + [A]_{i,2}[B + C]_{2,j} + \cdots\cdots \\
&\qquad + [A]_{i,n}[B + C]_{n,j} \\
&= [A]_{i,1}\left\{[B]_{1,j} + [C]_{1,j}\right\} + [A]_{i,2}\left\{[B]_{2,j} + [C]_{2,j}\right\} + \\
&\qquad \cdots\cdots + [A]_{i,n}\left\{[B]_{n,j} + [C]_{n,j}\right\} \\
&= \left\{[A]_{i,1}[B]_{1,j} + [A]_{i,2}[B]_{2,j} + \cdots\cdots + [A]_{i,n}[B]_{n,j}\right\} \\
&\quad + \left\{[A]_{i,1}[C]_{1,j} + [A]_{i,2}[C]_{2,j} + \cdots\cdots + [A]_{i,n}[C]_{n,j}\right\} \\
&= [AB]_{i,j} + [AC]_{i,j} \\
&= [AB + AC]_{i,j}
\end{aligned}$$

Therefore, $A(B + C)$ is equal to ....+.........

AB + AC

7. The formula

$$A(B + C) = AB + AC$$

is ....for matrices.
(correct/incorrect)

correct

8. The formula

$$A(B + C) = BA + CA$$

is ....for matrices.
(true/false)

false

9. The formula

$$A(B + C) = AB + CA$$

is .... .for matrices.
(true/false)

false

10. The formula

$$(B + C)A = BA + CA$$

is...................for matrices.
(correct/incorrect)

correct

11. Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

then

(a) $$A(B + C) = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix} = \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$$

- -

$$\begin{bmatrix} 3 & 0 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 18 & 24 \end{bmatrix}$$

---

and

(b) $$AB + AC = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix} + \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$$

$$= \begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$$

- -

$$\begin{bmatrix} 2 & -1 \\ 8 & 10 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 10 & 14 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 18 & 24 \end{bmatrix}$$

(c) Now

$$A(B + C) = \begin{bmatrix} 3 & 0 \\ 18 & 24 \end{bmatrix}, \text{ and } AB + AC = \begin{bmatrix} 3 & 0 \\ 18 & 24 \end{bmatrix}$$

therefore A(B + C) is .... to AB + AC.

equal

---

12. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

then

(a)

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b)

$$BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We notice that

$$AB \neq 0 \quad \text{but } BA = 0$$

This illustrates both the failure of the commutative law of multiplication and the breakdown of the law of cancellation.

13. Let

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{and } B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

then

(a)

$$AB = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{bmatrix}$$

and

(b)

$$BA = \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} ca - db & cb + da \\ -da - cb & -db + ca \end{bmatrix}$$

(c) We figured out that

$$AB = \begin{bmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{bmatrix}, \text{ and } BA = \begin{bmatrix} ca - db & cb + da \\ -da - cb & -db + ca \end{bmatrix}$$

It shows that AB and BA are ....
(equal/unequal)

equal

It is due to the special arrangement of a, b, c and d in two 2 x 2 matrices that the commutative law for multiplication holds good only in this arrangement.

---

14. Let

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

then

$$A \cdot A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \cdots\cdots$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

---

15. Let

$$A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$$

then

(a) $$AB = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} a^2 & 0 \\ 0 & -a^2 \end{bmatrix}$$

---

and

(b) $$BA = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} = \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix} = -\begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$$

$$\begin{bmatrix} -a^2 & 0 \\ 0 & a^2 \end{bmatrix} = -\begin{bmatrix} a^2 & 0 \\ 0 & -a^2 \end{bmatrix}$$

(c) Therefore $AB = -BA$

If $AB = -BA$, A and B are said to be anticommutative.

---

16. Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

then

$$AA - 5A + 2I = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} -5 \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} +2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix} + \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix} + \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$$

$$= \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix} = ...$$ (name of the matrix)

$$\begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix} + \begin{bmatrix} -10 & -5 \\ -20 & -15 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

It shows that A satisfied the equation $AA-5A + 2I = 0$

17. Let

$$\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -x & -14x & 7x \\ 0 & 1 & 0 \\ x & 4x & -2x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then

$$\begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(multiplication of two matrices)

$$\begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10x+2 & 5x \end{bmatrix}$$

---

18. If

$$\begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10x+2 & 5x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then x = ...

1/5

Unit IX: Powers of Matrices

Since matrix multiplication and ordinary multiplication are similar in many important ways, we can define the powers of a matrix in the ordinary way. Our definition shall lead us to the basic laws of exponents.

We define $A^2 = A \cdot A$, $A^3 = A \cdot A^2$, $A^4 = A \cdot A^3$, etc. That is, we make the inductive definition: $A^2 = A \cdot A$, $A^{n+1} = A \cdot A^n$.

It is also convenient to take $A^1 = A$ and $A^0 = I$.

---

1. Let A be any matrix. Then

$$A^2 \cdot A^3 = (A \cdot A)\ (A \cdot A \cdot A) = A \cdot A \cdot A \cdot A \cdot A = A^x$$

What does x equal? ....

x = 5

---

2. Let B be any matrix. Then

$$(B^2)^3 = (B \cdot B)^3 = (B \cdot B)\ (B \cdot B)\ (B \cdot B)$$
$$= B \cdot B \cdot B \cdot B \cdot B \cdot B = B^y$$

What does y equal? ....

y = 6

3. Let A be a matrix. Then

$$A^n \cdot A^m = A^{n+m}$$

COMMENT

It should be noted that the law of exponents applies only to powers of a single matrix A. We cannot conclude that

$$ABAB^2 = A^2B^3,$$

since this would involve rearrangement of factors. Such a formula can very well be false. Care must be taken to keep the factors in a product in correct order.

On the other hand, a formula like

$$ABA^2AA^4B^2A = ABA^7B^2A,$$

which involves only the law of exponents and not the rearrangement of factors, is true.

4. Let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$$

then

(a)
$$\begin{aligned} AAB &= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 16 \\ 9 & 36 \end{bmatrix}$$

(b)

$$\begin{aligned} ABA &= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 3 & 2 \\ 3 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 9 \\ 6 & 39 \end{bmatrix}$$

(c)

$$\begin{aligned} BAA &= \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 2 & -2 \\ 2 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -4 \\ 4 & 41 \end{bmatrix}$$

---

5. In the previous frame we figured out that

$$AAB = \begin{bmatrix} 9 & 16 \\ 9 & 36 \end{bmatrix}, \quad ABA = \begin{bmatrix} 6 & 9 \\ 6 & 39 \end{bmatrix}, \quad \text{and } BAA = \begin{bmatrix} 4 & -4 \\ 4 & 41 \end{bmatrix}$$

Therefore AAB, ABA and BAA are all ....

different
(not equal)

---

6. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

then

(a)

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

(b)

$$A^3 = A \cdot A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

$$= \quad ....$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

A matrix B is called nilpotent if some power of B is 0. Therefore in the above frame A is a nilpotent matrix, because $A^3 = 0$ and $A \neq 0$.

---

7. If

$$A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} \quad \text{then } A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore A is called ....matrix

---

nilpotent

8. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix} = \ldots\ldots\ldots\ldots$$

(name of the matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A$$

If $B^2 = B$, B is said to be an idempotent matrix. Therefore in the above frame A is an idempotent matrix because $A^2 = A$.

9. If

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \text{ then } A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Therefore A is called ....matrix

idempotent

C O M M E N T

Let A and B be matrices of the same size. Then

$$(2A)\,(3A) = 2((3A)A) = 2(3(A\cdot A)) = 6A^2$$

and

$$(2A)\,(3B) = 2((3A)\cdot B) = 2(3(AB)) = 6AB$$

Note that we continue to pay careful attention to the order in which the matrices come. 6AB must be distinguished from 6BA; only the numerical factors, and not the matrix factors, of a product may be rearranged.

---

10. Let A and B be the matrices of the same size. Then

$(6A)(3A)(B)(4) = \ldots\ldots\ldots\ldots\ldots\ldots$ (after multiplication)

$72A^2B$

---

11. Let A and B be the matrices of the same size. Then

$(A^3)(6A)(5B)(A^2) = \ldots\ldots\ldots\ldots\ldots\ldots$ (after multiplication)

$30A^4BA^2$

---

12. Let A and B be matrices of the same size, and let x be a real number. Then

$$x(AB) = (xA)B = A(xB).$$

Note that this formula permits free rearrangement of the numerical factors in a product.

13. Let A and B be two matrices of the same size and these do not commute, i.e., $AB \neq BA$. Then

$$(A+B)^3 = (A+B)(A+B)(A+B)$$

$$= (A+B)(A(A+B)+B(A+B))$$

$$= (A+B)(......+.....+.....+.....)$$

$$= A(A^2+AB+BA+B^2)+B(A^2+AB+BA+B^2)$$

$$= A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3$$

$$A^2+AB+BA+B^2$$

C O M M E N T

We have seen that in matrix algebra

$$(A+B)^3 = A^3+A^2B+ABA+AB^2+BA^2+BAB+B^2A+B^3$$

And there we must stop. If A and B were numbers we could add $A^2B+ABA+BA^2$ to get $3A^2B$, and $AB^2+BAB+B^2A$ to get $3AB^2$. But this involves a rearrangement of factors which is permissible for numbers, impermissible for matrices. This simple counter-example rules out the possibility of there being a binomial theorem for (non-commutative) matrices. Therefore the breakdown of the commutative law for multiplication for matrix algebra makes another difference between matrix algebra and ordinary algebra. In ordinary algebra where binomial theorem holds good $(a+b)^3$ would have been expanded very easily without multiplication, with the help of the binomial theorem as follows:

$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \ldots\ldots + b^n$

(where $n! = n(n-1)(n-2)\ \ldots\ldots\ 3\cdot2\cdot1$)

Using this binomial theorem

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

---

14. Let A and B be two non-commutative matrices of the same size. Then

$(A+3B)^2(A-2B) = (\ldots\ldots + \ldots\ldots + \ldots\ldots + \ldots\ldots)\ (A-2B)$

$= (A^2+3AB+3BA+9B^2)A + (A^2+3AB+3BA+9B^2)(-2B)$

$= \ldots\ldots + \ldots\ldots + \ldots\ldots + \ldots\ldots - \ldots\ldots - \ldots\ldots - \ldots\ldots - \ldots\ldots$

$A^2+3AB+3BA+9B^2$

$A^3+3ABA+3BA^2+9B^2A-2A^2B-6AB^2-6BAB-18B^3$

---

15. Let A and B be two non-commutative matrices of the same size. Then

$(A+B)(2A+3B)(A-3B) = (A+B)(2A(A-3B)+3B(A-3B))$

$= (A+B)(\ldots\ldots - \ldots\ldots + \ldots\ldots - \ldots\ldots)$

$= A(2A^2-6AB+3BA-9B^2)+B(2A^2-6AB+3BA-9B^2)$

$= \ldots\ldots - \ldots\ldots + \ldots\ldots - \ldots\ldots + \ldots\ldots - \ldots\ldots + \ldots\ldots - \ldots\ldots$

$2A^2-6AB+3BA-9B^2$

$2A^3-6A^2B+3ABA-9AB^2+2BA^2-6BAB+3B^2A-9B^3$

---

16. Let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

then

$$A^2-2A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} -2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} + \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \underset{\text{(name of the matrix)}}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

It shows that A satisfies the equation $A^2-2A = 0$

---

17. Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

then

$$A^2-3A+2I = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} -3 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} +2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} + \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} + \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \underset{\text{(name of the matrix)}}{\cdots\cdots\cdots\cdots}$$

$$\begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

---

18. We saw in the previous frame that

$$A^2 - 3A + 2I = 0$$

therefore A .... the equation $A^2-3A + 2I = 0$.
(satisfies/does not satisfy)

- -

satisfies

---

19. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

then

(a)

$$A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

- -

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(b)

$$A^3 = A\cdot A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

(c) Therefore

$$A^3-2A^2+I = \begin{bmatrix} 3 & 2 & 2 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} -2 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix} = ...$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

(d) It shows that $A^3-2A^2+I = 0$, therefore A satisfies the equation

....

$$A^3-2A^2+I = 0$$

20. Let

$$A = \begin{bmatrix} 1 & 2x \\ 2/x & 1 \end{bmatrix}$$

then

(a)

$$A^2-2A-3I = \begin{bmatrix} 1 & 2x \\ 2/x & 1 \end{bmatrix}\begin{bmatrix} 1 & 2x \\ 2/x & 1 \end{bmatrix} -2\begin{bmatrix} 1 & 2x \\ 2/x & 1 \end{bmatrix} -3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \ldots.. & \ldots.. \\ \ldots.. & \ldots.. \end{bmatrix} + \begin{bmatrix} \ldots.. & \ldots.. \\ \ldots.. & \ldots.. \end{bmatrix} + \begin{bmatrix} \ldots.. & \ldots.. \\ \ldots.. & \ldots.. \end{bmatrix}$$

$$= \begin{bmatrix} \ldots.. & \ldots.. \\ \ldots.. & \ldots.. \end{bmatrix} = \ldots\ldots\ldots$$

$$\begin{bmatrix} 5 & 4x \\ 4/x & 5 \end{bmatrix} + \begin{bmatrix} -2 & -4x \\ -4/x & -2 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(b) Therefore, $A^2-2A-3I = 0$.

It shows that A satisfies the equation......................

$$A^2-2A-3I = 0$$

No matter what the value of x the matrix A satisfies the equation $A^2-2A-3I = 0$, and therefore this quadratic equation has infinitely many distinct 2 x 2 matrices as roots.

COMMENT

We have seen that the law of exponents is the same for powers of a matrix as for powers of a number and that numerical factors in a product may be rearranged at will. Moreover, we know that the laws governing addition and subtraction of matrices are just like the laws governing addition and subtraction of numbers. It follows from all this that the algebraic rules for manipulating expressions which are made up out of sums of powers of a single matrix A, multiplied by arbitrary numerical coefficients, are just like the algebraic rules for manipulating ordinary polynomials. Thus, for instance,

$$(A+2I)^2 = A\cdot A+A\cdot 2I+2I\cdot A+2I\cdot 2I$$
$$= A^2+4A+4I$$

just as

$$(x+2)^2 = x^2+4x+4$$

and

$$(A-I)^2(A+I) = (A-I)(A^2-I)$$
$$= A^3-A^2-A+I$$

just as

$$(x-1)^2(x+1) = (x-1)(x^2-1)$$
$$= x^3-x^2-x+1$$

---

21. Factor the following polynomials in A into a product of first-degree polynomials in A:

(a) A^2-I = ...

$(A-I)(A+I)$

(b) A^3+3A^2+3A+I = ...

$(A+I)^3$

(c) $A^2-4A+4I$ = ...

$(A-2I)^2$

(d) $A^3-A^2-2A = A(A^2-A-2I) =$ ....

$A(A-2I)(A+I)$

(e) $A^4-5A^2+4I = (A^2-4I)(A^2-I) =$ ....

$(A-2I)(A+2I)(A-I)(A+I)$

---

22. A matrix satisfying the polynomial equations

(a) $X^2-5X+6I = 0$ is

$(X-3I)(X-2I) = 0$

$X =$..... or $X =$.....

$X = 3I$ or $X = 2I$

(b) $X^3-6X^2+12X-8I = 0$ is

....

$X = 2I$

Unit X: Determinants

So far you have learned addition, subtraction, and multiplication of matrices. Now before proceeding to division, we will discuss an operation applicable to the elements of a square matrix that leads to a scalar value known as the DETERMINANT of the matrix.

A determinant is a polynomial of the elements of a square matrix. The notation for the determinant of the matrix A is $|A|$. Determinants are defined only for square matrices--the determinant of a non-square matrix is undefined and does not exist. The determinant of a matrix is a scalar. Obtaining the value of $|A|$ by adding the appropriate products of the elements of A is referred to as evaluating the determinant, expanding the determinant or reducing the determinant.

1. The determinant of a 1 x 1 matrix

$$A = \begin{bmatrix} a \end{bmatrix}$$

is written as

$$\ldots\ldots = |a|$$

$|A|$

Note that the determinant of a 1 x 1 matrix is the value of its sole element.

---

2. The determinant of a 2 x 2 matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

is written as

$$|B| = \begin{vmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{vmatrix}$$

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

3. The determinant of a 2 x 2 matrix consists of the product of the diagonal terms minus the product of the off-diagonal terms. Hence in general

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= \quad ad - \dots\dots$$

bc

4. The determinant of a 2 x 2 matrix

$$|A| = \begin{vmatrix} 3 & 7 \\ 17 & 20 \end{vmatrix} = \dots\dots - \dots\dots = -59$$

3x20 - 7x17

5. The determinant of a 2 x 2 matrix

$$|B| = \begin{vmatrix} 10 & 1.6 \\ 5 & -9.2 \end{vmatrix} = -\dots\dots - \dots\dots = -\dots\dots$$

-92 -8 = -100

6. Evaluating the determinant of a 2 x 2 matrix is known as the evaluation of a second-order determinant. Evaluating the determinant of a 3 x 3 matrix is known as the evaluation of a ....determinant.

---

third-order

---

7. A third-order determinant can be evaluated as a linear function of three second-order determinants derived from it. Their coefficients are elements of a row (or column) of the main determinant, each product being multiplied by +1 or -1. For example the evaluation of the determinant of a 3 x 3 matrix

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

based on the elements of the first row, a_{11}, a_{12}, and a_{13}, is

$$|A| = a_{11}(+1)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(+1)\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(\ldots\ldots-\ldots\ldots) - a_{12}(\ldots\ldots-\ldots\ldots) + a_{13}(\ldots\ldots-\ldots\ldots)$$

(evaluation of 2nd-order determinant)

$$= \ldots\ldots\ldots-\ldots\ldots\ldots-\ldots\ldots\ldots+\ldots\ldots\ldots+\ldots\ldots\ldots-\ldots\ldots\ldots$$

$(a_{22}a_{33}-a_{23}a_{32})$; $(a_{21}a_{33}-a_{23}a_{31})$; $(a_{21}a_{32}-a_{22}a_{31})$

$a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}$

---

8. In the previous frame second-order determinants

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}; \quad \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}; \text{ and } \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

which are obtained from the determinant $|A|$, are known as minors of the elements a_{11}, a_{12} and a_{13} respectively, where a_{11}, a_{12} and a_{13} are the elements of the 1st row of the determinant $|A|$.

---

9. The evaluation of the determinant of a 3 x 3 matrix

$$|A| = \begin{vmatrix} 4 & 5 & 6 \\ 2 & 3 & 1 \\ 8 & 9 & 7 \end{vmatrix}$$

$$= 4(+1) \begin{vmatrix} 3 & 1 \\ 9 & 7 \end{vmatrix} +5(-1) \begin{vmatrix} 2 & 1 \\ 8 & 7 \end{vmatrix} +6(+1) \begin{vmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{vmatrix}$$

$$= 4(\ldots\ldots-\ldots\ldots) -5(\ldots\ldots-\ldots\ldots)+6(\ldots\ldots-\ldots\ldots)$$

$$= 4\text{x}12-5\text{x}6+6\text{x}-\ldots\ldots=-18$$

- -

$$\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$$

(21-9); (14-8); (18-24)

-6

---

10. In the previous frame:

(a) Second-order determinant

$$\begin{vmatrix} 3 & 1 \\ 9 & 7 \end{vmatrix}$$

is the minor of the element .... in the determinant $|A|$

- -

4

(b) The minor of the element 5 in the first row and second column of the determinant $|A|$ is

$$\begin{vmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 \\ 8 & 7 \end{vmatrix}$$

(c) What is the minor of the element 6 in the first row and 3rd column ?

$$\begin{vmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$$

---

11. The determinant

$$|A| = \begin{vmatrix} 4 & 5 & 6 \\ 2 & 3 & 1 \\ 8 & 9 & 7 \end{vmatrix}$$

can also be expanded based on the elements of the second row, 2, 3 and 1. The expansion will be

$$|A| = 2(-1)\begin{vmatrix} 5 & 6 \\ 9 & 7 \end{vmatrix} + 3(+1)\begin{vmatrix} 4 & 6 \\ 8 & 7 \end{vmatrix} + 1(-1)\begin{vmatrix} 4 & 5 \\ 8 & 9 \end{vmatrix}$$

$$= -2(\ldots\ldots - \ldots\ldots) + 3(\ldots\ldots - \ldots\ldots) - 1(\ldots\ldots - \ldots\ldots)$$

$$= (-2x-19)+(3x-20)-(1x-4)=-18$$

(35-54); (28-48); (36-40)

---

12. The expansion of the determinant

$$|A| = \begin{vmatrix} 4 & 5 & 6 \\ 2 & 3 & 1 \\ 8 & 9 & 7 \end{vmatrix}$$

based on the elements of the 3rd row, 8, 9 and 7 will be

$$|A| = 8(+1)\begin{vmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{vmatrix} + 9(-1)\begin{vmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{vmatrix} + 7(+1)\begin{vmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{vmatrix}$$

$$= 8(\text{.....} - \text{.....}) - 9(\text{.....} - \text{.....}) + 7(\text{.....} - \text{.....}) = -18$$

$$\begin{vmatrix} 5 & 6 \\ 3 & 1 \end{vmatrix}; \begin{vmatrix} 4 & 6 \\ 2 & 1 \end{vmatrix}; \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix}$$

(5-18); (4-12); (12-10)

COMMENT

The method of expanding a determinant which we have learned in the previous frames by using elements of any row can also be applied to the elements of any column. No matter by what row or column the expansion is made, the value of the determinant is the same. This method of expanding a determinant is known as expansion by the elements of a row (or column) or as expansion by minors. The (+1) and (-1) factors are decided on according to the following rule: if A is written in the form

$A = \left[a_{ij}\right]$, the product of a_{ij} and its minor in the expansion of the determinant $|A|$ is multiplied by $(-1)^{i+j}$. Once a row or column is decided on and the sign calculated for the product of the first element therein with its minor, the signs for the following products alternate from plus to minus and minus to plus. The product of $(-1)^{i+j}$ to the corresponding minor is known as the cofactor of a_{ij} in $|A|$.

---

13. The expansion of a fourth-order determinant is an extension of the expansion of a third-order determinant which you have learned. Thus in the expansion of a fourth-order determinant

$$|A| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$$

by the first row

(a) the minor of the element 4 is

$$\begin{vmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{vmatrix}$$

$$\begin{vmatrix} 5 & 6 & 7 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{vmatrix}$$

3

(b) The product of the element 4 with its minor is multiplied by

$(-1)^{\dots+\dots} = -1$

$(-1)^{1+4}$.

---

14. The expansion of the determinant

$$|A| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$$

by the second column is

$$|A| = 2(-1)^{\dots} \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} + 6(+1) \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} + 10(-1) \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

$$+ 14(+1) \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

$$= -2\left\{5x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix} - 7x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix} + 8x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix}\right\} + 6\left\{1x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix} - 3x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix}\right.$$

$$\left. + 4x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix}\right\} - 10\left\{1x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix} - 3x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix} + 4x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix}\right\} + 14\left\{1x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix}\right.$$

$$\left. - 3x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix} + 4x\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix}\right\}$$

$=-10(176-180)+14(144-156)-16(135-143)+6(176-180)-18(144-156)$

$+24(135-143)-10(112-120)+30(80-104)-40(75-91)+14(84-88)$

$-42(60-72)+56(55-63)=0$

$$2(-1)^3\begin{vmatrix}5&7&8\\9&11&12\\13&15&16\end{vmatrix} + 6(+1)\begin{vmatrix}1&3&4\\9&11&12\\13&15&16\end{vmatrix} + 10(-1)\begin{vmatrix}1&3&4\\5&7&8\\13&15&16\end{vmatrix} + 14(+1)\begin{vmatrix}1&3&4\\5&7&8\\9&11&12\end{vmatrix}$$

$$= -2\left\{5x\begin{vmatrix}11&12\\15&16\end{vmatrix} - 7x\begin{vmatrix}9&12\\13&16\end{vmatrix} + 8x\begin{vmatrix}9&11\\13&15\end{vmatrix}\right\} + 6\left\{1x\begin{vmatrix}11&12\\15&16\end{vmatrix} - 3x\begin{vmatrix}9&12\\13&16\end{vmatrix} + 4x\begin{vmatrix}9&11\\13&15\end{vmatrix}\right\}$$

$$-10x\left\{1\begin{vmatrix}7&8\\15&16\end{vmatrix} - 3x\begin{vmatrix}5&8\\13&16\end{vmatrix} + 4x\begin{vmatrix}5&7\\13&15\end{vmatrix}\right\} + 14\left\{1x\begin{vmatrix}7&8\\11&12\end{vmatrix} - 3x\begin{vmatrix}5&8\\9&12\end{vmatrix} + 4x\begin{vmatrix}5&7\\9&11\end{vmatrix}\right\}$$

COMMENT

The method of expanding a determinant used in the previous frames for expanding determinants of third-order and fourth-order, can be used for the expansion of determinants of any size. The determinant of an n x n matrix $A = \left[a_{ij}\right]$ is obtained, by considering the elements of any one row (or column), as follows: multiply each element, aij, of any row (or column) by its minor, $\left|M_{ij}\right|$, the determinant derived from $\left|A\right|$ by erasing out the row and column containing a_{ij}; multiply the product by $(-1)^{i+j}$; add the signed products and their sum is the determinant $\left|A\right|$. This expansion is used recurrently when n is large, i.e., each $\left|M_{ij}\right|$ is expanded by the same procedure.

---

15. Let

$$A = \begin{bmatrix}3&2&4\\-1&1&5\\6&7&10\end{bmatrix}; \text{ then}$$

(a)

$$A' = \begin{bmatrix}\cdots\cdots&\cdots\cdots&\cdots\cdots\\\cdots\cdots&\cdots\cdots&\cdots\cdots\\\cdots\cdots&\cdots\cdots&\cdots\cdots\end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 6 \\ 2 & 1 & 7 \\ 4 & 5 & 10 \end{bmatrix}$$

(b) Also

$$|A| = 3x \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} -2x \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} +4x \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix}$$

$$= (3x-25)-(2x-40)+(4x-13)=-47$$

$$\& \; |A'| = 3x \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} +1x \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} +6x \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix}$$

$$= (3x-25)-8+(6x6)=-47$$

therefore $|A|$ and $|A'|$ are ....

$$|A| = 3x \begin{vmatrix} 1 & 5 \\ 7 & 10 \end{vmatrix} -2x \begin{vmatrix} -1 & 5 \\ 6 & 10 \end{vmatrix} +4x \begin{vmatrix} -1 & 1 \\ 6 & 7 \end{vmatrix}$$

$$|A'| = 3x \begin{vmatrix} 1 & 7 \\ 5 & 10 \end{vmatrix} +1x \begin{vmatrix} 2 & 7 \\ 4 & 10 \end{vmatrix} +6x \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix}$$

equal

Note: Notice here $|A| = |A'|$. It is also true for any other square matrix of any size.

---

16. Let

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 7 & -2 & 4 \\ 6 & 9 & 10 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 7 & -2 & 4 \\ 3 & 1 & 5 \\ 6 & 9 & 10 \end{bmatrix}$$

where the first two rows of A are interchanged to form B.

(a)

$$|A| = 3x \begin{vmatrix} \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot \end{vmatrix} -1x \begin{vmatrix} \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot \end{vmatrix} +5x \begin{vmatrix} \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot \end{vmatrix}$$

$$= 3(-.....-.....)-1(.....-.....)+5(.....+.....) = 161$$

$$|A| = 3x\begin{vmatrix} -2 & 4 \\ 9 & 10 \end{vmatrix} -1x\begin{vmatrix} 7 & 4 \\ 6 & 10 \end{vmatrix} +5x\begin{vmatrix} 7 & -2 \\ 6 & 9 \end{vmatrix}$$

$$= 3(-20-36)-(70-24)+5(63+12)$$

(b)

$$|B| = 7x\begin{vmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{vmatrix} +2x\begin{vmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{vmatrix} +4x\begin{vmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{vmatrix}$$

$$= 7(\dots\dots-\dots\dots)+2(\dots\dots-\dots\dots)+4(\dots\dots-\dots\dots) = -161$$

$$|B| = 7x\begin{vmatrix} 1 & 5 \\ 9 & 10 \end{vmatrix} +2x\begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix} +4x\begin{vmatrix} 3 & 1 \\ 6 & 9 \end{vmatrix}$$

$$= 7(10-45)+2(30-30)+4(27-6)$$

(c) From (a)

$$|A| = 161$$

and from (b)

$$|B| = -161$$

therefore $|A| = (\dots\dots)\,|B|$

(-)
minus

Note: Notice that by changing two rows of the matrix A, the sign of its determinant is changed. It is true in general that interchanging two rows of a determinant changes its sign.

17. The expansion of

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is

$$|A| = a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(\ldots\ldots-\ldots\ldots)-a_{12}(\ldots\ldots-\ldots\ldots)+a_{13}(\ldots\ldots-\ldots\ldots)$$

$$= \ldots\ldots-\ldots\ldots-\ldots\ldots+\ldots\ldots+\ldots\ldots-\ldots\ldots$$

$$= \ldots\ldots$$

$$a_{11}(a_{12}a_{33}-a_{13}a_{32})-a_{12}(a_{11}a_{33}-a_{13}a_{31})+a_{13}(a_{11}a_{32}-a_{12}a_{31})$$

$$= a_{11}a_{12}a_{33}-a_{11}a_{13}a_{32}-a_{12}a_{11}a_{33}+a_{12}a_{13}a_{31}+a_{13}a_{11}a_{32}-a_{13}a_{12}a_{31}$$

$$= 0$$

Note: Notice that here $|A| = 0$. It can also be seen in another way: if the first two rows of $|A|$ are interchanged then the value of the determinant is unaltered but the sign is changed, that is, $|A| = -|A|$ so that $|A| = 0$. It is true for general that if two rows (or columns) of a determinant are the same the determinant is zero.

18. Let

$$A = \begin{bmatrix} 3 & 12 & 18 \\ 2 & 1 & 5 \\ 6 & 7 & 9 \end{bmatrix}; \text{ then}$$

(a)

$$|A| = \begin{vmatrix} 3 & 12 & 18 \\ 2 & 1 & 5 \\ 6 & 7 & 9 \end{vmatrix} = 3\text{x}\begin{vmatrix} 1 & 5 \\ 7 & 9 \end{vmatrix} - 12\text{x}\begin{vmatrix} 2 & 5 \\ 6 & 9 \end{vmatrix} + 18\text{x}\begin{vmatrix} 2 & 1 \\ 6 & 7 \end{vmatrix}$$

$$= 3(\ldots\ldots-\ldots\ldots)-12(\ldots\ldots-\ldots\ldots)+18(\ldots\ldots-\ldots\ldots)$$

$$= 210$$

$$3(9-35)-12(18-30)+18(14-6)$$

(b) also

$$|A| = (3)\begin{vmatrix} 1 & 4 & 6 \\ 2 & 1 & 5 \\ 6 & 7 & 9 \end{vmatrix} = 3\left\{1\text{x}\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix} - 4\text{x}\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix} + 6\begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix}\right\}$$

$$= 3\left\{(9-35)-4(18-30)+6(14-6)\right\} = 210$$

$$1\text{x}\begin{vmatrix} 1 & 5 \\ 7 & 9 \end{vmatrix} - 4\text{x}\begin{vmatrix} 2 & 5 \\ 6 & 9 \end{vmatrix} + 6\text{x}\begin{vmatrix} 2 & 1 \\ 6 & 7 \end{vmatrix}$$

Note: If any scalar is a factor of a row (or column) it is also a factor of the determinant. As in the above case 3 was the factor of the first row and it is also the factor of the determinant.

19. Let

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{11} & 3a_{12} & 3a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (3)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Therefore the value of $|A| = 0$, because the first two rows of the right hand side determinant are ....

the same or equal

Note: It is true in general that if one row (or column) of a determinant is a multiple of another row, the determinant is zero, as factoring out the multiple reduces the determinant to having two rows (or columns) which are the same and hence it is zero.

20. Let

$$|A| = \begin{vmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

then $|A|$ can be written as

$$|A| = 0 \times \begin{vmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

because zero is factor of the first row. Therefore the value of $|A|$ is ....

---

zero

---

21. If a determinant has a row (or column) of zeros the value of the determinant is zero, because .... is here a factor of one row (or column) and hence a factor of the determinant, which is therefore zero.

---

zero

---

22. Let

$$A = \begin{bmatrix} 3 & 0 & 15 \\ -9 & 12 & 21 \\ 6 & 27 & -3 \end{bmatrix} \text{; then } |A| = \ldots\ldots x \begin{bmatrix} 1 & 0 & 15 \\ -3 & 12 & 21 \\ 2 & 27 & -3 \end{bmatrix}$$

$$= \ldots\ldots x \ldots\ldots x \ldots\ldots x \begin{bmatrix} 1 & 0 & 5 \\ -3 & 4 & 7 \\ 2 & 9 & -1 \end{bmatrix} = 3^{\cdots} x \begin{bmatrix} 1 & 0 & 5 \\ -3 & 4 & 7 \\ 2 & 9 & -1 \end{bmatrix}$$

---

3 ; 3 x 3 x 3 ; 3^3

---

23. Let A be an n x n matrix and y is a scalar which is multiplied to each element of A, then $|yA| = y^{\cdots\cdot}|A|$

y^n

---

24. Let

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 5 & -4 & 4 \\ 9 & 1 & 7 \end{vmatrix} = 1(-28-4)-2(35-36)+3(5+36) = 93$$

then multiplying the first row by 3 and adding it to the third row we get

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 5 & -4 & 4 \\ 9+\ldots\ldots & 1+\ldots\ldots & 7+\ldots\ldots \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 5 & -4 & 4 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix}$$

$$= 1(-64-28)-2(80-48)+3(35+48) = 93$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 5 & -4 & 4 \\ 9+3 & 1+6 & 7+9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 5 & -4 & 4 \\ 12 & 7 & 16 \end{vmatrix}$$

Note: Notice that when three times the first row is added to the 3rd row the value of $|A|$ remains the same. It is true in general that adding to one row (or column) of a determinant any multiple of another row (or column) does not affect the value of the determinant.

25. We have seen that adding a multiple of a row (or column) to another row (or column) does not affect the value of a determinant, but adding a row (or column) to a multiple of another row (or column) is not the same thing and leads to a different result. Therefore if

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

then adding row 2 to y times row 1 we get

$$\begin{vmatrix} ya_{11}+a_{21} & \ldots+\ldots & \ldots+\ldots \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

which is equal to $y|A|$ which is different from $|A|$.

$ya_{12}+a_{22}$; $ya_{13}+a_{23}$

C O M M E N T

So far we have learned the following properties of determinants:

1. The determinant of the transpose of a matrix is the same as the determinant of the matrix: $|A'| = |A|$
2. If two rows (or columns) of a matrix are interchanged, the determinant of the matrix changes sign.
3. If two rows (or columns) of a matrix A are identical, then $|A| = 0$.
4. If all the elements of a row (or of a column) of a square matrix A be multiplied by an element k, the determinant of the matrix is multiplied by k.

5. If two rows (or columns) of a matrix A are proportional, then $\left| A \right| = 0$.
6. If a matrix A has a row (or column) of zeros the value of its determinant is zero: $\left| A \right| = 0 \times \left| A \right|$.
7. If A is an n x n matrix and k is a scalar then $\left| kA \right| = k^n \left| A \right|$.
8. Adding to one row (or column) of a determinant any multiple of another row (or column) does not affect the value of the determinant.

   These properties can be applied in endless variation in expanding determinants and thus the expansion becomes much simpler.

---

26. The expansion of the determinant

$$\left| A \right| = \begin{vmatrix} 1 & 5 & 3 \\ 2 & 11 & 7 \\ 4 & 21 & 15 \end{vmatrix} = 1(165-147)-5(30-28)+3(42-44) = 2.$$

$\left| A \right|$ can also be expanded as follows:

If -5 times column 1 is added to column 2 and
-3 times column 1 is added to column 3 we get

$$|A| = \begin{vmatrix} 1 & 0 & \dots \\ 2 & 1 & \dots \\ 4 & 1 & \dots \end{vmatrix} = 1\text{x} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2$$

0

1

3

Unit XI: Inversion of Matrices

1. In the previous units of this program we have studied addition, subtraction, and multiplication of matrices and the properties of determinants. It should not be surprising that our next aim is to study the division of matrices. In ordinary algebra, division is necessary to solve equations like ax=b. There is a similar problem with matrices, viz., solving equations like AX=B. Since the commutative law of multiplication breaks down for matrices, AX need ....XA.

not equal

---

2. Therefore AX=B and XA=B are .... problems.

different

---

3. If we solve AX=B and YA=B, then X need ....Y

not equal

---

4. Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

then AX = B becomes

$$\begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} X = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

---

5. Let

$$X = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Then

$$AX = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \underset{\text{(name of the matrix)}}{\cdots\cdots\cdots\cdots\cdots\cdots}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = B$$

Notice that $AX = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = B$, therefore $X = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ satisfies the equation AX=B

---

6. When $X = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ satisfies the equation AX=B,

then $X = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ is the .... of the equation AX=B.

---

solution

7. Now let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and $X = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

then

$XA = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \cdots\cdots\cdots\cdots\cdots$ (name of the matrix)

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = B$

Notice that $X = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ is the solution of the equation XA=B.

---

8. Now we have seen that the solution of the equation

AX=B is $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ and the solution of the equation

XA=B is $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$. It shows that the value of X in both cases is ....
(same/different)

---

different (not same)

Notice that it is due to the fact that $AX \neq XA$.

9. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$, and $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $AX = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$$\begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix}$$

Notice that if $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the solution of the equation $AX = B$, then $\begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix}$ should be equal to $\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$, but it is not possible whatever value you give to a, b, c, and d; because the second row will always be zero. Thus the proposed equation has no solution.

COMMENT

We have seen in the previous frame that for particular matrices A and B, the equation AX = B has no solution. Thus, division by the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is impossible, just as 'division by zero' is impossible in ordinary algebra. Therefore, division by some matrices is impossible.

---

10. Let $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} X = \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$

Multiplying both sides by the matrix $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$ we get

$$\begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} X = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$$

or $\begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} X = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$$

Notice that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and $IX = X$, therefore $X = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ is the solution of the equation $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} X = \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$

---

11. Let $A = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$

Then the equation $AX = B$ becomes

$$\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$$

Multiplying through (on the left) by a matrix $\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$, the inverse of A, which is called A^{-1},

we get $\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} X = \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$

or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 2 & 11 \\ -1 & -9 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$$

Notice that the value of $X = \begin{bmatrix} 2 & 11 \\ -1 & -9 \end{bmatrix}$

COMMENT

Now we have seen that if we multiply through (on the left) the equation $AX = B$ by another matrix A^{-1} such that $A^{-1}A = I$, then we can solve the equation as follows:

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Thus $X = A^{-1}B$ is the solution to the equation $AX = B$

Since A^{-1} behaves very much like a reciprocal in ordinary algebra, we shall call it the reciprocal matrix of A or the inverse matrix of A.

Thus the problem of division of matrices is actually the problem of finding the reciprocal or inverse of a given matrix. Some matrices have reciprocal matrices, some do not.

To find out the inverse of a 2 x 2 we proceed as follows:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and its inverse $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if A has an inverse

or $\begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Therefore

$$\begin{aligned} ap + br &= 1 \\ cp + dr &= 0 \\ aq + bs &= 0 \\ cq + ds &= 1 \end{aligned}$$

Solving the above equations we get

$$p = \frac{d}{ad-bc}, \quad q = \frac{-b}{ad-bc}$$

$$r = \frac{-c}{ad-bc}, \quad \text{and } s = \frac{a}{(ad-bc)}$$

Therefore

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

12. In any 2 x 2 matrix, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \text{ then its inverse } A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

provided that $ad-bc \neq 0$.

$$\text{Because } AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \cdots\cdots\cdots\cdots\cdots \text{ (name of the matrix)}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Note: Notice that it is a general formula for finding the inverse of a 2 x 2 matrix. This formula tells us that a 2 x 2 matrix has an inverse if and only if ad-bc $\neq 0$. (ad-bc) is the determinant of A. It is true in general that A^{-1} does exist only if $|A|$ is non-zero. A square matrix is said to be singular when its determinant is zero and nonsingular when its determinant is non-zero.

---

13. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The matrix A has its inverse matrix

if .... - .... $\neq 0$

ad-bc

---

14. Let

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

(a) The element of the first row and first column of A^{-1} = $\dfrac{d}{\text{.......-........}}$

$\dfrac{d}{ad-bc}$

(b) The element of the first row and 2nd column of A^{-1} = $\dfrac{-b}{\text{...... - ......}}$

$\dfrac{-b}{ad-bc}$

15. If the inverse of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

equal $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

Then

(a) The inverse of the matrix $B = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix}$ is

B^{-1} is $\begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ -1/4 & 1/2 \end{bmatrix}$$

(b) The inverse of the matrix $C = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ is

$C^{-1} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$

$$\begin{bmatrix} -2/3 & 1 \\ 1/3 & 0 \end{bmatrix}$$

---

16. Let

$$A = \begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}$$

Does A have an inverse? .... (yes/no)

----

no

---

17. Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1/4 & 1/2 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & 1 \\ 0 & -5 \end{bmatrix}$

Then for solving AX = B, it becomes

$$A^{-1}AX = A^{-1}B$$

or $IX = A^{-1}B$ (as $A^{-1}A = I$)

or $X = A^{-1}B$ (as $IX = X$)

Therefore

$$X = \begin{bmatrix} 1 & -1 \\ -1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ -3/4 & -11/4 \end{bmatrix}$$

---

18. Let

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -2/3 & 1 \\ 1/3 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Solve the equation AX = B

$$X = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix} \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix} = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

$$X = \begin{bmatrix} -2/3 & 1 \\ 1/3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -5/3 & 4/3 \\ 1/3 & 1/3 \end{bmatrix}$$

19. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$AA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} = \dots\dots \text{ (name of the matrix)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Notice that AA = I, therefore A is its own inverse.

20. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = A^{-1}, \text{ and } B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

Then

Solve the equation AX = B

$$X = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

---

21. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

(a)

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} = \text{.....}$$

(name of the matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Notice that when $AB = I$, then A is the inverse of B, or $A = B^{-1}$

(b) and

$$BA = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} \end{bmatrix} = \text{.....}$$

(name
of th
matri

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Here $BA = I$, therefore B is the inverse of A, or $B = A^{-1}$

COMMENT

We have seen in the previous frame that $AB = I = BA$ which shows that A is the inverse of B and B is the inverse of A. If we put $B = A^{-1}$ in the above equation we get

$$AA^{-1} = I = A^{-1}A$$

Therefore A commutes with its inverse. It is true in general that every matrix commutes with its inverse provided the inverse exists and is similar to the ordinary algebra where $aa^{-1} = 1 = a^{-1}a$. Also if a matrix has an inverse, it has only one inverse; that is, this inverse is unique.

22. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Then

(a) $$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}$$

(b)

$$A - I = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$$

(c)

$$A^2 - 7I = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 12 \\ 18 & 24 \end{bmatrix}$$

(d)

$$(A^2-7I)-6(A-I) = \begin{bmatrix}0 & 12\\ 18 & 24\end{bmatrix} -6 \begin{bmatrix}0 & 2\\ 3 & 4\end{bmatrix}$$

$$= \begin{bmatrix}\cdots\cdots & \cdots\cdots\\ \cdots\cdots & \cdots\cdots\end{bmatrix} = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0$$

---

23. We saw in the previous frame that

$(A^2-7I)-6(A-I) = 0$

or $A^2-6A-I = 0$

Therefore the matrix A satisfies the equation

.... = 0

$A^2-6A-I = 0$

C O M M E N T

Once we have discovered an equation which is satisfied by a particular matrix, it can be used to find a reciprocal for the matrix. As the equation

$A^2-6A-I = 0$ can be written as

$A^2-6A = I$

or $A(A-6I) = I$

Then $(A-6I)$ must be the inverse of A.

24. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $(A-6I)$ is its inverse, then

$$A^{-1} = A-6I = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} -6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

---

25. If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

Then

$$A^{-1}A = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix} = \text{..........}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

---

26. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

and

(a) $A-I = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

(b) $$A^2-I = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

(c)

$$(A^2-I)-2(A-I) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} -2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \cdots\cdots$$

Therefore $A^2-I-2(A-I) = 0$

or $A^2-2A+I = 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

---

27. In the previous frame we figured out that A satisfied the equation

$$A^2-2A+I = 0$$

or $$A(-A+2I) = I$$

Therefore the inverse of A is

$A^{-1} = -\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots$

-A+2I

---

28. If $A^2-2A+I = 0$, then

$A^2 = 2A-I$

So $A^4 = (2A-I)\cdots$

$= 4A^2- 4A + I$.

2

$4A^2-4A+I$ can further be reduced by substituting $A^2= 2A-I$, getting

$4(2A-I)-4A+I$

$= 4A-3I$

---

29. If $A^4 = 4A-3I$, then

$A^8 = (4A-3I)^2$

$= \ldots\ldots - \ldots\ldots + \ldots\ldots$

$16A^2-24A+9I$

---

30. Polynomial $16A^2-24A+9I$ can be simplified further by substituting $A^2= 2A-I$, getting

$16(2A-I)-24A+9I$

$= \ldots\ldots\ldots - \ldots\ldots\ldots$

COMMENT

The polynomial equation satisfied by A is useful in doing other calculations involving A. As we obtained the value of A^3 without much calculation. Once the polynomial equation of a matrix is obtained, it is much easier to study all the other properties of the matrix.

Let A be an n x n matrix. Then A satisfies a polynomial equation of the form $A^k + c_{k-1}A^{k-1} + \ldots\ldots\ldots\ldots + c_0 I = 0$ of a degree k which is not more than n and with numerical coefficients $c_{k-1}, \ldots\ldots, c_0$.

Thus, a 2 x 2 matrix always satisfies either a linear or a quadratic polynomial equation. A 3 x 3 matrix always satisfies a cubic, quadratic, or linear equation. A 4 x 4 matrix always satisfies a polynomial equation of degree 4 at most, etc.

31. Let

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}, \text{ then } A^2 = \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$$

and

(a) $A - I = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} - \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$$

(b)

$$A^2-4I = \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix} -4 \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 6 \\ 2 & 0 \end{bmatrix}$$

(c)

$$A^2-4I-2(A-I) = \begin{bmatrix} 0 & 6 \\ 2 & 0 \end{bmatrix} -2 \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \ldots\ldots\ldots$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

---

32. If $A^2-4I-2(A-I) = 0$

Then $A^2-2A-2I = 0$

or $A^2-2A = 2I$

or $A(1/2)(A-2I) = I$

Therefore the inverse of A is

$A^{-1} = \ldots\ldots\ldots\ldots\ldots$

(1/2) (A-2I)

33. If $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ and $A^{-1} = (1/2)(A-2I)$, Then

$$A^{-1} = 1/2 \left\{ \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} -2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= 1/2 \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix} = \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$$

$$1/2 \begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix}$$

34. If $A^2 = 2(A+I)$, then squaring both sides we get

$A^4 =$ ....

and after substituting $A^2 = 2(A+I)$ in the right hand side it becomes

$A^4 =$ ....

$$4(A^2+2A+I)$$

$$= 4(4A+3I)$$

35. If

$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, then $A^4 = 4(4A+3I)$

$$\text{or } A^4 = 4 \left\{ 4 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= 4 \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix} = \begin{bmatrix} ..... & ..... \\ ..... & ..... \end{bmatrix}$$

$$4\begin{bmatrix}7 & 12\\4 & 7\end{bmatrix}=\begin{bmatrix}28 & 48\\16 & 28\end{bmatrix}$$

36. Let

$$A=\begin{bmatrix}2 & 1\\5 & -8\end{bmatrix}\text{ and }I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

(a) What multiple of I should be subtracted from A so that the element of the first row and first column of the resultant matrix may be zero? ......

2

(b) Therefore

$$A-2I=\begin{bmatrix}2 & 1\\5 & -8\end{bmatrix}-2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\ldots\ldots & \ldots\ldots\\\ldots\ldots & \ldots\ldots\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\5 & -10\end{bmatrix}$$

37. If $A=\begin{bmatrix}2 & 1\\5 & -8\end{bmatrix}$, then $A^2=\begin{bmatrix}9 & -6\\-30 & 69\end{bmatrix}$

(a) What multiple of I should be subtracted from A^2 so that the element of the first row and first column of the resultant matrix may be zero?

....

9

(b) Therefore

$$A^2-9I = \begin{bmatrix} 9 & -6 \\ -30 & 69 \end{bmatrix} -9 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & -6 \\ -30 & 60 \end{bmatrix}$$

38. Now

$$A^2-9I = \begin{bmatrix} 0 & -6 \\ -30 & 60 \end{bmatrix} \text{ and } A-2I = \begin{bmatrix} 0 & 1 \\ 5 & -10 \end{bmatrix}$$

(a) What multiple of (A-2I) should be subtracted from or added to (A^2-9I) so that all the elements of the resultant matrix may be zero?.......

6 times added

(b) Therefore

$$A^2-9I+6(A-2I) = \begin{bmatrix} 0 & -6 \\ -30 & 60 \end{bmatrix} +6 \begin{bmatrix} 0 & 1 \\ 5 & -10 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \cdots\cdots$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

39. When $A^2-9I+6(A-2I) = 0$

or $A^2+6A-21I = 0$

Then the matrix A satisfies the equation

....

$A^2+6A-21I = 0$

C O M M E N T

From the previous frames you must have got an idea of how to find an equation satisfied by a 2 x 2 matrix. To summarize it, let A be a 2 x 2 matrix and I a unit matrix of size 2. To find the equation which may be satisfied by A, first we calculate A^2. Then find a number "a such that A-aI is a matrix with element 0 in the upper left-hand corner. If it happens that A-aI is the zero matrix, we are done, for in this case A satisfied the linear polynomial equation $A-aI = 0$. If, however, any element of A-aI is different from 0, we continue the process by finding a number "b" such that A^2-bI is a matrix having 0 in the upper left-hand corner. We now have two matrices which have 0 in the upper left-hand corner; viz., A-aI and A^2-bI. Now, for 2 x 2 matrices, if $A-aI \neq 0$, then there exists a number "c" such that

$$(A^2-bI)-c(A-aI) = 0$$

which is the required equation.

40. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) To find a quadratic equation satisfied by the matrix A, first I should be multiplied by .... (number) and then subtracted from .... (matrix) giving

$$A\text{-}I = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$1,\ A,\ \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) After I is multiplied by 1 and subtracted from A, the next step will be to multiply I by .... (number) and then subtract it from A^2, where $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, giving

$$A^2\text{-}2I = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$2,\ \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

(c) If $A\text{-}I = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ and $A^2\text{-}2I = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

Then to find a quadratic equation satisfied by the matrix A, (A-I) should be multiplied by .... (number) and then subtracted from (A^2-2I), giving

$$A^2\text{-}2I\text{-}(A\text{-}I) = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \ldots\ldots\ldots\ldots$$

$1, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

(d) If $A^2-2I-(A-I) = 0$

or $A^2-A-I = 0$

Then this equation is satisfied by the matrix ....

A

41. If $A^2-A-I = 0$

or $A^2-A = I$

then the inverse of A is

$A^{-1} =$ ....

A-I

42. If $A^2-A-I = 0$

or $A^2=A+I$

then $A^4=$ ....

and after substituting $A^2= A+I$, it becomes

$A^4=$ ....

A^2+2A+I

$3A+2I$

43. If $A^4 = 3A+2I$

then $A^8 =$ ....

and after substituting $A^2 = A+I$, it becomes

$A^8 =$ ....

$9A^2+12A+4I$

$21A+13I$

44. Now if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } A^8 = 21A+13I$$

$$\text{then } A^8 = 21 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 13 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix}$$

45. Let

$$A = \begin{bmatrix} 1/3 & 7 \\ 1/7 & 3 \end{bmatrix}$$

(a) To find a quadratic equation satisfied by the matrix A, first I should be multiplied by .... and then subtracted from .... giving

$$A-(1/3)I = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

1/3, A, $\begin{bmatrix} 0 & 7 \\ 1/7 & 8/3 \end{bmatrix}$

(b) When I is multiplied by 1/3 and subtracted from A, the next step to find an equation satisfied by the matrix A will be to multiply I by .... and then subtract it from A^2 where $A^2 = \begin{bmatrix} 10/9 & 70/3 \\ 10/21 & 10 \end{bmatrix}$,

giving $A^2-(10/9)I = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$

- -

10/9 , $\begin{bmatrix} 0 & 70/3 \\ 10/21 & 80/9 \end{bmatrix}$

(c) If

$A-(1/3)I = \begin{bmatrix} 0 & 7 \\ 1/7 & 8/3 \end{bmatrix}$ and $A^2-(10/9)I = \begin{bmatrix} 0 & 70/3 \\ 10/21 & 80/9 \end{bmatrix}$

Then to find an equation satisfied by the matrix A, (A-(1/3)I) should be multiplied by .... and then subtracted from $(A^2-(10/9)I)$, giving

$A^2-(10/9)I-(10/3)(A-(1/3)I) = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \cdots\cdots\cdots$

- -

10/3

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

(d) If $A^2-(10/9)I-(10/3)(A-(1/3)I) = 0$

or $A^2-(10/3)A = 0$

then A satisfied the equation

.... = 0

$$A^2-(10/3)A = 0$$

Notice that A has no inverse as the equation which it satisfies has no term which contains I (alone).

$A = \begin{bmatrix} 1/3 & 7 \\ 1/7 & 3 \end{bmatrix}$ has no inverse is true also if we test it from our earlier formula as follows:

$$(1/3)(3)-(1/7)(7) = 0$$

Unit XII: Inversion of Matrices (continued)

In the previous unit you learned a procedure to find an equation satisfied by a 2 x 2 matrix A. This equation was used to find an inverse of the matrix A. The polynomial equation satisfied by A was also used to calculate A raised to a high power.

The above procedure can be applied to matrices of higher orders. This procedure is based upon a theorem which tells us that every n x n matrix A satisfies a polynomial equation

$$A^k + c_{k-1} A^{k-1} + \ldots\ldots + c_0 I = 0$$

with numerical coefficients, the degree k of this equation being at most n.

In this unit you will learn the procedure to find the least equation satisfied by a matrix A of any size and then find the inverse of A with the help of the least equation satisfied by A.

1. Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

then

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} = \dots\dots$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

---

2. If $A^2 = 0$, then the matrix A satisfies the equation

....

$$A^2 = 0$$

---

3. If $A^2 = 0$, then A^3 and A^4 also equal to zero. Therefore the matrix A satisfies the equation $A^2 = 0$ as well as the equations

.... = ... and .... = ...

$A^3 = 0, \quad A^4 = 0$

COMMENT

We noticed in the previous frames that the matrix A of order 4 satisfied equation of 4th degree as well as equations of 3rd degree and 2nd degree. Therefore the equation of the second degree is the equation of the lowest degree which is satisfied by the matrix A. It should be clear that the equation of lowest degree which a matrix satisfies is the equation which gives the most information about the matrix. There is one and only one such equation of least degree for each matrix. The polynomial of smallest possible degree which a matrix A satisfies is called the least equation satisfied by A.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Then } A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $A^3 = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Also

(a)

$$A-I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$$A^2-I = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c)

$$A^3-I = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(d)

$$(A^2-I)-2(A-I) = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} -2 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix} = \ldots\ldots$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Notice that $(A^2-I)-2(A-I) = 0$ or $A^2-2A + I = 0$ is satisfied by A.

(e)

$$(A^3-I)-3(A-I) = \begin{bmatrix} 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} -3 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix} = \cdots\cdots$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Here too $(A^3-I)-3(A-I) = 0$ or $A^3-3A + 2I = 0$ is satisfied by A.

(f) We saw that the matrix A of 3rd order satisfied the equations

$$A^2-2A + I = 0$$

and $$A^3-3A + 2I = 0$$

Therefore the least equation satisfied by A is

.... = ...

$$A^2-2A + I = 0$$

---

5. If $A^2-2A + I = 0$ or $2A-A^2 = I$, then the inverse of A is

$$A^{-1} = 2I-\cdots\cdots$$

A

6. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Then $A^{-1} = 2I-A$

or $A^{-1} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Let

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) What multiple of I should be subtracted from A so that the element of the first row and first column of the resultant matrix may be zero? ......

1

(b) Therefore

$A-I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

8\.

If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(a) What multiple of I should be subtracted from A^2 so that the element of the first row and first column of the resultant matrix may be zero? ......

2

(b) Therefore

$$A^2-2I = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} -2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

9\.

If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, then $A^3 = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

(a) What multiple of I should be subtracted from A^3 so that the element of the first row and first column of the resultant matrix may be zero? ....

4

(b) Therefore

$$A^3-4I = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} -4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

- -

$$\begin{bmatrix} 0 & 2 & 3 \\ 1 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

---

10. Now

$$A-I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and } A^2-2I = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

(a) What multiple of (A-I) should be subtracted from (A^2-2I) so that the resulting matrix has zeros in row one column two as well as in row one column one? ....

- -

1

(b) Therefore

$$(A^2-2I)-(A-I) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

- -

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

---

11. Also

$$A-I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \text{ and } A^3-4I = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

(a) What multiple of (A-I) should be subtracted from (A^3-4I) so that the resulting matrix has zeros in row one column two as well as in row one column one? ....

- -

2

(b) Therefore

$$(A^3-4I)-2(A-I) = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} -2 \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

- -

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

---

12. Now

$$(A^2-2I)-(A-I) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } (A^3-4I)-2(A-I) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) What multiple of $\left\{(A^2-2I)-(A-I)\right\}$ should be subtracted from $\left\{(A^3-4I)-2(A-I)\right\}$ so that the resulting matrix is the zero matrix? ....

- -

1

(b) Therefore

$$\left\{(A^3-4I)-2(A-I)\right\} - \left\{(A^2-2I)-(A-I)\right\} = \begin{bmatrix}0 & 0 & 1\\ 1 & -1 & -1\\ 0 & 1 & 0\end{bmatrix} - \begin{bmatrix}0 & 0 & 1\\ 1 & -1 & -1\\ 0 & 1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}\ldots\ldots & \ldots\ldots & \ldots\ldots\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\end{bmatrix} = \ldots\ldots\ldots$$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = 0$$

13. We have

$$\left\{(A^3-4I)-2(A-I)\right\} - \left\{(A^2-2I)-(A-I)\right\} = 0$$

or $A^3-A^2-A-I = 0$, which is the least equation satisfied by ....?

A

COMMENT

In the previous unit we gave a systematic procedure for finding the least equation satisfied by a 2 x 2 matrix.

Also in the previous frames you must have noticed that the same procedure was used for finding the least equations of 3 x 3 matrices. The preceding procedure, being general, can be applied to matrices of any size.

Let A be an n x n matrix. We first calculate A^2, A^3, ..., A^n. We then find numbers a_1, a_2, ..., a_n such that the matrices $B_1 = A-a_1I$, $B_2 = A^2-a_2I$, ..., $B_n = A^n-a_nI$ all have the number 0 in the upper left-hand corner. If $B_1 = 0$, we are done, for then $A-a_1I = 0$ is the least equation we are seeking. If not, we find a second set of numbers b_2, b_3, ..., b_n such that the matrices $C_2 = B_2-b_2B_1$, $C_3 = B_3-b_3B_1$, ..., $C_n = B_n-b_nB_1$ have zeros for elements $[C]1,2$ as well as for $[C]_{1,1}$.
If $C_2 = 0$, i.e., if $(A^2-a_2I)-b_2(A-a_1I) = 0$, we have the least equation satisfied by A; if not the process is continued.

---

14. Let

$$A = \begin{bmatrix} \sqrt{2} & 3 \\ -4 & 0 \end{bmatrix}$$

(a) To find the least equation satisfied by the matrix A, first I should be multiplied by .... and then subtracted from ...., giving

$$A-\sqrt{2}I = \begin{bmatrix} \sqrt{2} & 3 \\ -4 & 0 \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$\sqrt{2}$, A

$$\begin{bmatrix} 0 & 3 \\ -4 & -\sqrt{2} \end{bmatrix}$$

(b) When I is multiplied by $\sqrt{2}$ and subtracted from A, the resulting equation $A-\sqrt{2}I \ldots 0$, therefore the next step to find the least equation satisfied by
(= or ≠)
the matrix A will be to multiply I by 10 and add it to A^2, where

$$A^2 = \begin{bmatrix} -10 & 3\sqrt{2} \\ -4\sqrt{2} & -12 \end{bmatrix}, \text{ giving}$$

$$A^2+10I = \begin{bmatrix} -10 & 3\sqrt{2} \\ -4\sqrt{2} & -12 \end{bmatrix} + 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$\neq$

$$\begin{bmatrix} 0 & 3\sqrt{2} \\ -4\sqrt{2} & -2 \end{bmatrix}$$

(c) Now $A-\sqrt{2}I = \begin{bmatrix} 0 & 3 \\ -4 & -\sqrt{2} \end{bmatrix}$ and $A^2 + 10I = \begin{bmatrix} 0 & 3\sqrt{2} \\ -4\sqrt{2} & -2 \end{bmatrix}$

Then to find the least equation satisfied by the matrix A,

$(A-\sqrt{2}I)$ should be multiplied by .... and then subtracted from

$(A^2 + 10I)$, giving

$$A^2+10I-\sqrt{2}(A-\sqrt{2}I) = \begin{bmatrix} 0 & 3\sqrt{2} \\ -4\sqrt{2} & -2 \end{bmatrix} -\sqrt{2} \begin{bmatrix} 0 & 3 \\ -4 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \ldots\ldots$$

$\sqrt{2}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(d) If $A^2 + 10I - \sqrt{2}(A - \sqrt{2}I) = 0$

or $A^2 - \sqrt{2}A + 12I = 0$

Then the least equation satisfied by A is ....

$A^2 - \sqrt{2}A + 12I = 0$

15. If $A^2 - \sqrt{2}A + 12I = 0$

or $(1/12)\ (\sqrt{2}I - A)A = I$

then the inverse of A is

$A^{-1} = $ ....

$(1/12)\ (\sqrt{2}I - A)$

16. Let

$$A = \begin{bmatrix} \sqrt{2} & 3 \\ -4 & 0 \end{bmatrix}, \text{ then } A^{-1} = (1/12)\ (\sqrt{2}I - A)$$

$$\text{or } A^{-1} = 1/12 \left\{ \sqrt{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \sqrt{2} & 3 \\ -4 & 0 \end{bmatrix} \right\}$$

$$= 1/12 \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$1/12 \begin{bmatrix} 0 & -3 \\ 4 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & -1/4 \\ 1/3 & \sqrt{2}/12 \end{bmatrix}$$

17. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

(a) To find the least equation satisfied by the matrix A, first I should be multiplied by .... and then subtracted from ...., giving

$$A-I = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

1, A

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

(b) When I is multiplied by 1 and subtracted from A, the resulting equation $A-I \neq 0$, therefore the next step to find the least equation satisfied by A will be to multiply I by .... and subtract it from A^2,

where $A^2 = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 1 & 1 \\ 7 & 3 & 6 \end{bmatrix}$ giving

$$A^2-3I = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 1 & 1 \\ 7 & 3 & 6 \end{bmatrix} -3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

3

$$\begin{bmatrix} 0 & 1 & 3 \\ 2 & -2 & 1 \\ 7 & 3 & 3 \end{bmatrix}$$

(c) Also multiply I by .... and subtract it from A^3

where $A^3 = \begin{bmatrix} 10 & 4 & 9 \\ 5 & 2 & 4 \\ 22 & 9 & 19 \end{bmatrix}$ giving

$$A^3-10I = \begin{bmatrix} 10 & 4 & 9 \\ 5 & 2 & 4 \\ 22 & 9 & 19 \end{bmatrix} -10 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

10

$$\begin{bmatrix} 0 & 4 & 9 \\ 5 & -8 & 4 \\ 22 & 9 & 9 \end{bmatrix}$$

(d) Now

$A^2-3I = \begin{bmatrix} 0 & 1 & 3 \\ 2 & -2 & 1 \\ 7 & 3 & 3 \end{bmatrix}$ and $A^3-10I = \begin{bmatrix} 0 & 4 & 9 \\ 5 & -8 & 4 \\ 22 & 9 & 9 \end{bmatrix}$

The next step to find the least equation is to get a second matrix (of the above two) with zero as element in row one, column two. Therefore (A^2-3I) should be multiplied by .... and subtracted from (A^3-10I), giving

(continued)

$$A^3-10I-4(A^2-3I) = \begin{bmatrix} 0 & 4 & 9 \\ 5 & -8 & 4 \\ 22 & 9 & 9 \end{bmatrix} -4 \begin{bmatrix} 0 & 1 & 3 \\ 2 & -2 & 1 \\ 7 & 3 & 3 \end{bmatrix} = \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix}$$

4

$$\begin{bmatrix} 0 & 0 & -3 \\ -3 & 0 & 0 \\ -6 & -3 & -3 \end{bmatrix}$$

(e) Now

$$A-I = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} \text{ and } A^3-10I-4(A^2-3I) = \begin{bmatrix} 0 & 0 & -3 \\ -3 & 0 & 0 \\ -6 & -3 & -3 \end{bmatrix}$$

To get the zero matrix (of the above two), (A-I) should be multiplied by .... and added to $A^3-10I-4(A^2-3I)$

giving

$$A^3-10I-4(A^2-3I) + 3(A-I) = \begin{bmatrix} 0 & 0 & -3 \\ -3 & 0 & 0 \\ -6 & -3 & -3 \end{bmatrix} +3 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix} = ...$$

3

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

(f) The equation $A^3-10I-4(A^2-3I) + 3(A-I) = 0$

or $A^3-4A^2 + 3A-I = 0$ is the ....satisfied by A.

least equation

18. If $A^3-4A^2 + 3A-I = 0$ is the least equation satisfied by A, then the inverse of A is

A^{-1} = ...

$A^2-4A + 3I$

19. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

then

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

= ....

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

---

20. If

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ then } A^2 = I$$

Therefore the inverse of A is

$$A^{-1} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Notice that $A^2 = I$ implies that A is its own inverse.

C O M M E N T

We saw in the previous frames how to calculate the least equation satisfied by a matrix. Once we know the least equation satisfied by a matrix A, we can easily see whether or not A has a reciprocal.

If the final coefficient c_0 in the least equation satisfied by A is different from zero, then A has a reciprocal as we can see from the following:

We may write the least equation satisfied by A in the form

$$A^k + c_{k-1}A^{k-1} + \ldots\ldots\ldots + c_1A = -c_0I$$

or $(A^{k-1} + c_{k-1}A^{k-2} + \ldots\ldots\ldots\ldots + c_1I)A = -c_0I$

Since the numerical coefficient c_0 is different from zero, we can divide both sides of the equation by the number $-c_0$ and find that

$$-(1/c_0)\ (A^{k-1} + c_{k-1}A^{k-2} + \ldots\ldots\ldots + c_1I)A = I$$

This makes it plain that the matrix

$$\left[-(1/c_0)\ (A^{k-1} + c_{k-1}A^{k-2} + \ldots\ldots\ldots + c_1I)\right]$$

is a reciprocal of A.

If the coefficient c_0 in the least equation satisfied by A is equal to zero, then A does not have a reciprocal.

21. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

and let $A^{-1} = B$. To find the least equation satisfied by B, I should be multiplied by .... and added to B, giving

$$B + 5I = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} \neq 0$$

- -

5

$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$$

---

22. $B + 5I \neq 0$, therefore to find the least equation satisfied by B, now I should be multiplied by .... and subtracted from B^2

where $B^2 = \begin{bmatrix} 31 & -12 \\ -18 & 7 \end{bmatrix}$ giving

$$B^2 - 31I = \begin{bmatrix} 31 & -12 \\ -18 & 7 \end{bmatrix} - 31 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

31

$$\begin{bmatrix} 0 & -12 \\ -18 & -24 \end{bmatrix}$$

23. We have $B + 5I = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$ and $B^2 - 31I = \begin{bmatrix} 0 & -12 \\ -18 & -24 \end{bmatrix}$

To find the least equation satisfied by B, $(B + 5I)$ should be multiplied by .... and added to $B^2 - 31I$, giving

$$B^2 - 31I + 6(B+5I) = \begin{bmatrix} 0 & -12 \\ -18 & -24 \end{bmatrix} + 6 \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix} = \cdots\cdots\cdots\cdots$$

6

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

24. The equation

$$B^2 - 31I + 6(B+5I) = 0$$

$$\text{or} \quad B^2 + 6B - I = 0$$

is satisfied by B. Therefore the inverse of B is

$B^{-1} =$ ....

B + 6I

25. Now

$$B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \text{ and } B^{-1} = B+6I$$

Therefore

$$B^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

26. In the previous frames we figured out that

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

$$\text{and if } A^{-1} = B, \text{ then } B^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$$\text{or } (A^{-1})^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \text{........................}$$

(name of matrix)

A

That is, the inverse of A^{-1} is A.

---

27. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

then

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

- - - - - - - - - - - - - - - - - - - -

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

---

28. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Then $A^{-1} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$

- - - - - - - - - - - - - - - - - - - -

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

---

29. If $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Then $B^{-1} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$

- - - - - - - - - - - - - - - - - - - -

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

30. If $A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

then

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

31. If $AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

then $(AB)^{-1} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

32. Now $B^{-1}A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ and $(AB)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

Therefore $(AB)^{-1}$ is .... to $B^{-1}A^{-1}$
(equal/unequal)

equal

We saw that if A and B both have inverses, then so does their product AB. Also it is true for general, i.e., given a finite number of (square) matrices of the same size, each of which has an inverse, the inverse of the product of the matrices is equal to the product of their inverses written in the reverse order.

---

33. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, find A^{-1} and B^{-1}.

$A^{-1} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

- -

$A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

---

34. If $A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

then

$B^{-1}A^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

- -

$\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$

---

35. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$

$(AB)^{-1} = \begin{bmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{......} \end{bmatrix}$

$\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$

---

36. $(AB)^{-1} = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} = B^{-1}A^{-1}$ shows that

the inverse of (AB) is equal to the product of the inverse of ....

and the inverse of ....

B , A

---

37. If $(AB)^{-1}$ is the inverse of (AB), then

$(AB)^{-1}(AB) = \text{.........} = (AB)\ (AB)^{-1}$

I

---

38. If A satisfies the equation $A^2-2A+3I = 0$, does the inverse of A exist? ....
(yes or no)

Yes

---

39. If A satisfies the equation $A^2-2A + 3I = 0$, what is the inverse of A?

A^{-1} = ...

(1/3) (2I-A)

---

40. Let A and B be n x n matrices and assume B^{-1} exists. Then

$$(BAB^{-1})^k = (BAB^{-1})\ (BAB^{-1})\ .\ .\ .\ (BAB^{-1})$$

⟵ k of them ⟶

These k factors of the right hand side can be regrouped by taking out B from the next factor and including it in the previous factor. Therefore:

$$(BAB^{-1})^k = (BAB^{-1}B)\ (AB^{-1}B)\ .\ .\ .\ (AB^{-1}B)\ (............)$$

⟵ k factors ⟶

(write last factor)

AB^{-1}

---

41. The right hand side of

$$(BAB^{-1})^k = (BAB^{-1}B)\ (AB^{-1}B)\ .\ .\ .\ (AB^{-1}B)\ (AB^{-1})$$

⟵ k factors ⟶

can be simplified further by substituting $B^{-1}B = I$, leaving

$$(BAB^{-1})^k = B \qquad A^k \qquad .....$$

B^{-1}

Unit XIII: Inversion of Matrices (other method)

The inverse of a square matrix can also be derived from properties of the cofactors of elements of its determinant. A property of cofactors important for developing the inverse of a matrix is that the sum of products of the elements of one row (or column) with the cofactors of the elements of another row (or column) is zero.

1. The cofactors of the elements of the first row of the determinant

$$|A| = \begin{vmatrix} 1 & 4 & -8 \\ 2 & 5 & 9 \\ 3 & 6 & 10 \end{vmatrix} \quad \text{are}$$

$$\begin{vmatrix} 5 & 9 \\ 6 & 10 \end{vmatrix} = -4 \;;\; (-1)\begin{vmatrix} 2 & 9 \\ 3 & 10 \end{vmatrix} = \ldots\ldots \;;\; \begin{vmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{vmatrix} = -3$$

$$7 \;;\; \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$$

2. In the previous frame the cofactors of the elements 1, 4 and -8 are -4, 7 and -3. Therefore the determinant of A is

$-4x1 + 7x4 - \ldots\ldots x \ldots\ldots = 48$

-3 x -8

3. The cofactors of the elements of the first row of the determinant

$$|A| = \begin{vmatrix} 1 & 4 & -8 \\ 2 & 5 & 9 \\ 3 & 6 & 10 \end{vmatrix} = 48$$

are -4, 7 and -3.

(a) The sum of products of the elements of the 2nd row with the cofactors of the elements of the 1st row is:

2x-4 + ...x..... + ...x-..... = ...

5x7 + 9x-3 ; 0

(b) The sum of products of the elements of the 3rd row with the cofactors of the elements of the 1st row is:

.....x-..... + ...x..... + ...x-..... = ...

3x-4 + 6x7 + 10x-3 = 0

(c) The cofactors of the elements of the second row are:

$$(-1)\begin{vmatrix} 4 & -8 \\ 6 & 10 \end{vmatrix} = -88\ ;\ \begin{vmatrix} \text{.....} & \text{.....} \\ \text{.....} & \text{.....} \end{vmatrix} = 34\ ;\ \text{and } (-1)\begin{vmatrix} \text{....} & \text{....} \\ \text{....} & \text{....} \end{vmatrix} = \text{....}$$

$$\begin{vmatrix} 1 & -8 \\ 3 & 10 \end{vmatrix} \quad ; (-1) \begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} = 6$$

(d) The sums of products of the elements of the 1st, second and 3rd row with the cofactors of the elements of the second row are:

1x-88 + 4x34 + (-8)x6 = ...

....x-88 + ...x34 + ...x6 = ...

....x-88 + ...x34 + ...x6 = ...

0

2x-88 + 5x34 + 9x6 = 48

3x-88 + 6x34 + 10x6 = 0

(e) The cofactors of the elements of the 3rd row are:

$$\begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} = \cdots ; (-1) \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} = \cdots ; \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} = \cdots$$

$$\begin{vmatrix} 4 & -8 \\ 5 & 9 \end{vmatrix} = 76; \ (-1) \begin{vmatrix} 1 & -8 \\ 2 & 9 \end{vmatrix} = -25; \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = -3$$

(f) The sums of products of the elements of the 1st, second and 3rd row with the cofactors of the elements of the third row are:

.....x76 + ...x-25 + ...x-3 = ...

.....x76 + ...x-25 + ...x-3 = ...

.....x76 + ...x-25 + ...x-3 = ...

1x76 + 4x-25 + (-8)x-3 = 0

2x76 + 5x-25 + 9 x-3 = 0

3x76 + 6x-25 + 10 x-3 = 48

4. In frames 1, 2 and 3 we have seen that the sum of products of the elements of one row of A with the cofactors of the elements of another row of A is zero. Also the cofactors of the elements of A can be represented in a matrix B

$$\begin{bmatrix} -4 & 7 & -3 \\ -88 & 34 & 6 \\ 76 & -25 & -3 \end{bmatrix} = B \quad \text{where } A = \begin{bmatrix} 1 & 4 & -8 \\ 2 & 5 & 9 \\ 3 & 6 & 10 \end{bmatrix}$$

(a) Each element of matrix B is the cofactor of the corresponding element of matrix A. Therefore 34 is the cofactor of ....

(b) The transpose of B is called the adjugate, or the adjoint, matrix of A. Therefore adjugate matrix of A is

$$B' = \begin{bmatrix} \cdots\cdot & \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot & \cdots\cdot \end{bmatrix}$$

$$\begin{bmatrix} -4 & -88 & 76 \\ 7 & 34 & -25 \\ -3 & 6 & -3 \end{bmatrix}$$

(c) The inverse of A is calculated as $\frac{1}{|A|} \times B'$, where $|A| = 48$. Therefore

$$A^{-1} = \frac{1}{|A|} \times B' = \frac{1}{\cdots} \begin{bmatrix} \cdots\cdot & \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot & \cdots\cdot \end{bmatrix}$$

$$\frac{1}{48} \begin{bmatrix} -4 & -88 & 76 \\ 7 & 34 & -25 \\ -3 & 6 & -3 \end{bmatrix}$$

4

(d)

$$A^{-1}A = \frac{1}{48}\begin{bmatrix} -4 & -88 & 76 \\ 7 & 34 & -25 \\ -3 & 6 & -3 \end{bmatrix}\begin{bmatrix} 1 & 4 & -8 \\ 2 & 5 & 9 \\ 3 & 6 & 10 \end{bmatrix} = \frac{1}{48}\begin{bmatrix} \cdots\cdot & \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot & \cdots\cdot \\ \cdots\cdot & \cdots\cdot & \cdots\cdot \end{bmatrix} = \cdots$$

$$\frac{1}{48}\begin{bmatrix} 48 & 0 & 0 \\ 0 & 48 & 0 \\ 0 & 0 & 48 \end{bmatrix} = I$$

---

5. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 4 & 5 \\ 10 & 5 & 9 \end{bmatrix}$$

(a) The cofactors of each element of A are:

1st row 11 , -4 , -10

2nd row ..... , ... , ...

3rd row -3 , -3 , 6

1, 7, -5

(b) $|A| = 3\text{x}\ldots\ldots + 1\text{x-}\ldots\ldots + 2\text{x-}\ldots\ldots = \ldots\ldots$

$3\text{x}11 + 1\text{x-}4 + 2\text{x-}10 = 9$

(c) The matrix B of the cofactors of all the elements of A is

$$B = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$ and therefore adjugate of A is

$$\ldots\ldots\ldots = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 11 & -4 & -10 \\ 1 & 7 & -5 \\ -3 & -3 & 6 \end{bmatrix}; \; B' = \begin{bmatrix} 11 & 1 & -3 \\ -4 & 7 & -3 \\ -10 & -5 & 6 \end{bmatrix}$$

(d) Therefore

$$A^{-1} = \frac{1}{--} \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$\frac{1}{9} \begin{bmatrix} 11 & 1 & -3 \\ -4 & 7 & -3 \\ -10 & -5 & 6 \end{bmatrix}$$

6. Let $A = \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix}$

then

(a) $|A| = \ldots\ldots$

11

(b) The cofactor matrix of the elements of $A = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$\begin{bmatrix} 7 & -5 \\ -2 & 3 \end{bmatrix}$

(c) The adjugate matrix of $A = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$\begin{bmatrix} 7 & -2 \\ -5 & 3 \end{bmatrix}$

(d) and therefore the inverse of $A = A^{-1} = \frac{1}{---} \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$$\frac{1}{11}\begin{bmatrix} 7 & -2 \\ -5 & 3 \end{bmatrix}$$

C O M M E N T

We have learned that if A is a square, nonsingular matrix its inverse, A^{-1}, has the following properties.

(1) The inverse commutes with A, both products being the identity matrix: $A^{-1}A = I = AA^{-1}$.

(2) The inverse matrix is nonsingular.

(3) The inverse of A^{-1} is A: $(A^{-1})^{-1} = A$.

(4) The inverse of a transpose is the transpose of the inverse:

$$(A')^{-1} = (A^{-1})'.$$

(5) The inverse of a product is the product of the inverses taken in reverse order, provided the inverse exist: $(AB)^{-1} = B^{-1}A^{-1}$.

Unit XIV: Vectors

1. In the previous units we have studied square matrices as well as rectangular matrices of different sizes, but a matrix may consist of only one row or one column. This form of the matrix is called a vector. A matrix of one row and two columns will be called a row .... of size 2.

vector

2. A matrix of one row and three columns will be called a row .... of size 3.

vector

3. A matrix of size 1 x n will be called a row vector of size ....

n

4. A matrix of 2 rows and one column will be called a column .... of size 2.

vector

5. A matrix of size 3 x 1 will be called a column .... of size ....

---

vector, 3

---

6. A matrix of size n x 1 will be called .... vector of size ....

---

column, n

---

7. The following matrix $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

is a column .... of size ....

---

vector, 3

---

8. The following matrix $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

is a .... of size ....

---

row vector, 3

---

9. $\underline{v} = \begin{bmatrix} 1, 2, 3 \end{bmatrix}$ is an example of a row vector of size ....

3

---

10. $\underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an example of a column vector of size ....

2

---

11. Two row vectors, or two column vectors, are said to be equal if and only if corresponding components of the vector are equal. Thus the vectors $\underline{v} = \begin{bmatrix} 3, 4 \end{bmatrix}$ and $\underline{u} = \begin{bmatrix} 3, 4 \end{bmatrix}$ are .... (equal/unequal)

---

equal

---

12. If $\underline{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

then the vectors $\underline{u}$ and $\underline{v}$ are .... (equal/unequal)

---

equal

13. Vectors $\underline{u} = \begin{bmatrix} 1, & 2, & 3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are ....
(equal/unequal)

----

unequal

---

14. Let $\underline{u} = \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1, & v_2, & v_3 \end{bmatrix}$ be two row vectors.

Then their sum $\underline{u} + \underline{v}$ will be component-wise addition as follows:

$$\underline{u} + \underline{v} = \begin{bmatrix} u_1 + v_1, & u_2 + v_2, & u_3 + v_3 \end{bmatrix}$$

If $\underline{u} = \begin{bmatrix} 1,2,3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 4, & 5, & 6 \end{bmatrix}$

Then $\underline{u} + \underline{v} = \begin{bmatrix} \ldots.., & \ldots.., & \ldots.. \end{bmatrix}$

$\begin{bmatrix} 5, & 7, & 9 \end{bmatrix}$

---

15. Let

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Then $\underline{u} + \underline{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$ is a vector of size ....

3

16. Let

$$\underline{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \text{ and } \underline{v} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\text{Then } \underline{u} + \underline{v} = \begin{bmatrix} \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$$

---

17. Let $\underline{u} = \begin{bmatrix} -2, & 5, & 2 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 3, & -1, & 4 \end{bmatrix}$

Then $\underline{u} + \underline{v} = \begin{bmatrix} \cdots\cdots, & \cdots\cdots, & \cdots\cdots \end{bmatrix}$

and $\underline{v} + \underline{u} = \begin{bmatrix} \cdots\cdots, & \cdots\cdots, & \cdots\cdots \end{bmatrix}$

and therefore $\underline{u} + \underline{v}$ and $\underline{v} + \underline{u}$ are ....
(equal/uneaual)

$\begin{bmatrix} 1, & 4, & 6 \end{bmatrix}$

$\begin{bmatrix} 1, & 4, & 6 \end{bmatrix}$

equal

---

18. Let

$$\underline{u} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \text{ and } \underline{v} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$$

$$\text{Then } \underline{u} + \underline{v} = \begin{bmatrix} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{bmatrix}$$

$$\text{and } \underline{v} + \underline{u} = \begin{bmatrix} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{bmatrix}$$

and therefore $\underline{u} + \underline{v}$ and $\underline{v} + \underline{u}$ are ....
(equal/unequal)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ equal}$$

---

19. The sum of two row vectors of the same size 3 is the row vector of size ....

3

---

20. The sum of two column vectors of the same size 4 is the .... vector of size ....

---

column, 4

---

21. The sum of two n-component vectors (either row or column) is obtained by component-wise addition, and yields another n-component ....

vector

COMMENT

Observe that we do not add vectors unless they are both row or both column vectors, having the same number of components.

Also we have seen that the order in which vectors are added does not matter; that is,

$$\underline{u} + \underline{v} = \underline{v} + \underline{u}$$

where $\underline{u}$ and $\underline{v}$ are both row or both column vectors.

Therefore commutative law of addition holds true in the case of vectors also.

---

22. The multiplication of a number a times a vector $\underline{v}$ is defined by component-wise multiplication of a times the components of $\underline{v}$. For the three-component row vector we have

$$a\underline{v} = a \begin{bmatrix} v_1, v_2, v_3 \end{bmatrix} = \begin{bmatrix} av_1, av_2, av_3 \end{bmatrix}$$

and for the three-component column vector we have

$$a\underline{u} = a \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix}$$

---

23. If $\underline{v} = \begin{bmatrix} 1, & -4, & 7, & 9 \end{bmatrix}$ then

$5\underline{v} = [\ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots]$

$\begin{bmatrix} 5, & -20, & 35, & 45 \end{bmatrix}$

---

24. If $\underline{u} = \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix}$, then

$$(-1)\underline{u} = (-1)\begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix}$$
$$= [\ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots]$$

$\begin{bmatrix} -u_1, & -u_2, & -u_3 \end{bmatrix}$

---

25. If $\underline{u}$ is any vector then its negative $-\underline{u}$ is the vector $(-1)\underline{u}$. Therefore if

$\underline{u} = \begin{bmatrix} u_1, & u_2, & u_3, & u_4 \end{bmatrix}$, then

$-\underline{u} = (-1)\begin{bmatrix} u_1, & u_2, & u_3, & u_4 \end{bmatrix} = [\ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots]$

$\begin{bmatrix} -u_1, & -u_2, & -u_3, & -u_4 \end{bmatrix}$

---

26. Let $\underline{u} = \begin{bmatrix} 5, & 6, & 7 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 1, & 2, & 3 \end{bmatrix}$, then

$\underline{u}-\underline{v} = \begin{bmatrix} 5, & 6, & 7 \end{bmatrix} - \begin{bmatrix} 1, & 2, & 3 \end{bmatrix} = [\ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots]$

$$\begin{bmatrix} 4, & 4, & 4 \end{bmatrix}$$

---

27. Let

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and } \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{then}$$

$$\underline{u}-\underline{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \text{............} \\ \text{............} \\ \text{............} \end{bmatrix}$$

- -

$$\begin{bmatrix} u_1-v_1 \\ u_2-v_2 \\ u_3-v_3 \end{bmatrix}$$

---

28. Let

$$\underline{u} = \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}, \text{ and } \underline{w} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}. \quad \text{Then}$$

$$\underline{u}+3\underline{v}-5\underline{w} = \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix} +3 \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} -5 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix} + \begin{bmatrix} \text{.....} \\ \text{.....} \\ \text{.....} \end{bmatrix} + \begin{bmatrix} \text{.....} \\ \text{.....} \\ \text{.....} \end{bmatrix} = \begin{bmatrix} \text{.....} \\ \text{.....} \\ \text{.....} \end{bmatrix}$$

- -

$$\begin{bmatrix} -9 \\ 15 \\ 21 \end{bmatrix} , \begin{bmatrix} -5 \\ -15 \\ -20 \end{bmatrix}$$

$$\begin{bmatrix} -4 \\ 8 \\ 7 \end{bmatrix}$$

---

29. Let $\underline{u} = \begin{bmatrix} 7, & 0, & -5 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 2, & -3, & 1 \end{bmatrix}$, and $\underline{w} = \begin{bmatrix} 1, & 0, & -1 \end{bmatrix}$. Then

$2\underline{u}-3\underline{v}+5\underline{w} = 2 \begin{bmatrix} 7, & 0, & -5 \end{bmatrix} -3 \begin{bmatrix} 2, & -3, & 1 \end{bmatrix} + 5 \begin{bmatrix} 1, & 0, & -1 \end{bmatrix}$

=[...., ..., ...] + [...., ..., ...] + [...., ..., ...]

=[...., ..., ...]

$$\begin{bmatrix} 14, & 0, & -10 \end{bmatrix} + \begin{bmatrix} -6, & 9, & -3 \end{bmatrix} + \begin{bmatrix} 5, & 0, & -5 \end{bmatrix}$$

$$\begin{bmatrix} 13, & 9, & -18 \end{bmatrix}$$

---

30. Is the sum $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

possible ....
(yes/no)

no

31. A vector whose components are zero is the zero vector.

The vectors

$$\underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \underline{0} = \begin{bmatrix} 0, & 0, & 0 \end{bmatrix}$$

are three-component ....

---

zero vectors

---

32. Let

$$\underline{u} = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} \text{ , then}$$

$$\underline{u}+\underline{0} = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix}$$

---

33. Let $\underline{v} = \begin{bmatrix} v_1, & v_2, & v_3 \end{bmatrix}$, then

$$\underline{v} + \underline{0} = \begin{bmatrix} v_1, & v_2, & v_3 \end{bmatrix} + \begin{bmatrix} 0, & 0, & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cdots\cdots, & \cdots\cdots, & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} v_1, & v_2, & v_3 \end{bmatrix}$$

34. If $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, then

$u_1 =$ ....., $u_2 =$ ....., $u_3 =$ .....

- -

2, -3, -1

---

35. If

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is multiplied by zero, we get

$$0 \cdot \underline{v} = 0 \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{bmatrix}$$

- -

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

---

36. Let

$$A = \begin{bmatrix}3 & 4 & 7\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \text{ which is equal to the row vector } \begin{bmatrix}\ldots\ldots, & \ldots\ldots, & \ldots\ldots\end{bmatrix}$$

$$\text{and } B = \begin{bmatrix}2 & 0 & 0\\ 6 & 0 & 0\\ 1 & 0 & 0\end{bmatrix} \text{ which is equal to the column vector } \begin{bmatrix}\ldots\ldots\\ \ldots\ldots\\ \ldots\ldots\end{bmatrix}$$

Therefore

$$AB = \begin{bmatrix}3 & 4 & 7\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \begin{bmatrix}2 & 0 & 0\\ 6 & 0 & 0\\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}3, & 4, & 7\end{bmatrix} \begin{bmatrix}2\\ 6\\ 1\end{bmatrix}$$

- -

$$\begin{bmatrix}3, & 4, & 7\end{bmatrix}$$

$$\begin{bmatrix}2\\ 6\\ 1\end{bmatrix}$$

37.

$$\begin{bmatrix}6, & 12, & 3\end{bmatrix} \begin{bmatrix}4\\ 6\\ 9\end{bmatrix} = \ldots\ldots \times \ldots\ldots + \ldots\ldots \times \ldots\ldots + \ldots\ldots \times \ldots\ldots = 123$$

- -

$6 \times 4 + 12 \times 6 + 3 \times 9$

---

38. Let $\underline{u} = \begin{bmatrix} 2, & 6, & 1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$

Then $\underline{u}\,\underline{v} = \begin{bmatrix} 2, & 6, & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$ = ... + ... + ... = ...

$6 + 24 + 7 = 37$

39.

Let $\underline{u} = \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Then $\underline{uv} = \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ = ... + ... + ...

$u_1v_1 + u_2v_2 + u_3v_3$

---

40. We figured out in the previous frames that the product of a row vector and a column vector (both of the same size) is a ....
(number/vector/matrix)

---

number

41. We have seen that the multiplication of a row vector by a column vector (both of the same size) is similar to the multiplication of two matrices, and the vectors are conformable for multiplication because the number of columns of the row vector is equal to the number of .... of the column vector.

rows

42. Two row vectors of the same size .... conformable for multiplica-
(are/are not)
tion because the number of columns of the first vector ....
(is/ is not)
equal to the number of rows of the second vector.

are not

is not

43. Let $\underline{u} = \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1, & v_2, & v_3 \end{bmatrix}$

Then $\underline{u}\,\underline{v} = \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix} \begin{bmatrix} v_1, & v_2, & v_3 \end{bmatrix}$

Does it make any sense? ....
(yes/no)

No

44. Two column vectors of the same size .... conformable for
(are/are not)
multiplication.

are not

---

45. Let

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and } \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Then $\underline{u}\ \underline{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Does it make any sense? .... (yes/no)

- -

no

---

46. The multiplication of a row vector by a column vector is called a minor product or a scalar product.

The following product

$$\begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

.... a minor product.
(is/is not)

- -

is

---

47. We learned that a matrix can be transposed by changing rows into columns and columns into rows. The same principle can be used for transposing row and column vectors. Therefore after transposing a row vector it will become a .... vector and a column vector will become a .... vector

---

column

row

48. Let u and v be two column vectors of equal size, then the multiplication of the transposed of vector u by the vector v .... (is/is not) conformable for multiplication.

---

is

49. Let $\underline{u} = [u_1, u_2, u_3, u_4]$ and $\underline{v} = [v_1, v_2, v_3, v_4]$. Also $\underline{v}'$ is the transposed vector of v. Then

$$\underline{u}\,\underline{v}' = [u_1, u_2, u_3, u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \ldots\ldots + \ldots\ldots + \ldots\ldots + \ldots\ldots$$

$u_1v_1+u_2v_2+u_3v_3+u_4v_4$

50. Let $\underline{u} = \begin{bmatrix} u_1, & u_2, & u_3, & u_4 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1, & v_2, & v_3, & v_4 \end{bmatrix}$. Also $\underline{u}'$ is the transposed of $\underline{u}$. Then

$$\underline{v}\,\underline{u}' = \begin{bmatrix} v_1, & v_2, & v_3, & v_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \ldots\ldots + \ldots\ldots + \ldots\ldots + \ldots\ldots$$

$v_1u_1+v_2u_2+v_3u_3+v_4u_4$

---

51. Let $\underline{u} = \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\underline{u}', \underline{v}'$ their transposes.

Then

$$\underline{u}\,\underline{v} = \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \ldots\ldots + \ldots\ldots + \ldots\ldots$$

and $\underline{v}'\underline{u}' = \begin{bmatrix} v_1, & v_2, & v_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \ldots\ldots + \ldots\ldots + \ldots\ldots$

Therefore $\underline{u}\,\underline{v}$ and $\underline{v}'\underline{u}'$ .... equal.
(are/are not)

$u_1v_1+u_2v_2+u_3v_3$

$v_1u_1+v_2u_2+v_3u_3$

are

It is true in general that the minor product of two vectors is equal to the product of their transposes in reverse order.

---

52. The second kind of product of two vectors is called the major product of two vectors. The major product of two vectors is the product of a column vector by a row vector. The following product

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1, & v_2, & v_3, & v_4 \end{bmatrix}$$

is a .... product of two vectors.
(minor/major)

major

Notice in major product two vectors need not to be of the same size.

---

53. In a major product, two vectors of different sizes, the first a column vector and the second a row vector.................... conformable for multiplication.
(are/are not)

are

54. The major product of two vectors can be performed in the same way as the product of two matrices, i.e., multiplying row by column. The major product of the following two vectors will be

$$\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \begin{bmatrix} 1, & 4, & 2, & 6 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} 3 & 12 & 6 & 18 \\ 1 & 4 & 2 & 6 \\ 5 & 20 & 10 & 30 \end{bmatrix}$$

---

55. Multiply the following two vectors

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{bmatrix} v_1, & v_2, & v_3 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \\ u_4v_1 & u_4v_2 & u_4v_3 \end{bmatrix}$$

---

56. Let $\underline{v} = \begin{bmatrix}1, & 2, & 3\end{bmatrix}$. The minor product of $\underline{v}$ and its transpose $\underline{v}'$ is

$$\underline{v}\,\underline{v}' = \begin{bmatrix}\ldots\ldots, & \ldots\ldots, & \ldots\ldots\end{bmatrix} \begin{bmatrix}\ldots\ldots\\ \ldots\ldots\\ \ldots\ldots\end{bmatrix} = 1^2 + 2^2 + 3^2 = 14$$

(write two vectors)

$$\begin{bmatrix}1, & 2, & 3\end{bmatrix} \qquad \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

57. Let $\underline{v} = \begin{bmatrix}v_1, & v_2, & v_3\end{bmatrix}$. The minor product of $\underline{v}$ and its transpose $\underline{v}'$ is

$$\underline{v}\,\underline{v}' = \begin{bmatrix}v_1, & v_2, & v_3\end{bmatrix} \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \ldots\ldots + \ldots\ldots + \ldots\ldots$$

$$v_1^2 + v_2^2 + v_3^2$$

58. The minor product of a vector by its transpose is simply the sum of .... of the elements of the vector.

squares

59. Let $\underline{v} = \begin{bmatrix}1, & 2, & 3\end{bmatrix}$. The major product of $\underline{v}$ and its transpose $\underline{v}'$ will be

$$\underline{v}'\,\underline{v} = \begin{bmatrix}\ldots\ldots\\ \ldots\ldots\\ \ldots\ldots\end{bmatrix} \begin{bmatrix}\ldots\ldots & \ldots\ldots & \ldots\ldots\end{bmatrix} = \begin{bmatrix}1^2 & 2 & 3\\ 2 & 2^2 & 6\\ 3 & 6 & 3^2\end{bmatrix}$$

(write two vectors)

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1, & 2, & 3\end{bmatrix}$$

60. Let $\underline{v} = \begin{bmatrix}v_1, & v_2, & v_3\end{bmatrix}$. The major product of $\underline{v}$ and its transpose $\underline{v}'$ will be

$$\underline{v}'\,\underline{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\begin{bmatrix}v_1, & v_2, & v_3\end{bmatrix} = \begin{bmatrix}\cdots\cdots & \cdots\cdots & \cdots\cdots\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\end{bmatrix}$$

$$\begin{bmatrix}v_1^2 & v_1v_2 & v_1v_3\\ v_2v_1 & v_2^2 & v_2v_3\\ v_3v_1 & v_3v_2 & v_3^2\end{bmatrix}$$

C O M M E N T

A square matrix in which the corresponding elements of corresponding rows and columns are equal is called a symmetric matrix. As we saw in the previous frame that the major product of a row vector $\underline{v}$ and its transpose $\underline{v}'$,

$$\underline{v}'\,\underline{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\begin{bmatrix}v_1, & v_2, & v_3\end{bmatrix} = \begin{bmatrix}v_1^2 & v_1v_2 & v_1v_3\\ v_2v_1 & v_2^2 & v_2v_3\\ v_3v_1 & v_3v_2 & v_3^2\end{bmatrix}$$

has the elements of the first row equal to the corresponding elements of the first column, elements of the 2nd row equal to the corresponding elements of the 2nd column and the elements of the 3rd row equal to the corresponding elements of the 3rd column. In general the major product of a vector and its transpose is a symmetric matrix.

61. Let

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$ Then the major product of $\underline{u}$ and its transpose is

$$\underline{u}\,\underline{u}' = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix} = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

and the resultant matrix is a .... matrix.
(kind of matrix)

$$\begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2^2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3^2 \end{bmatrix}$$

symmetric

---

62. The vector being a special case of the matrix, the commutative law of multiplication which does not hold for the product of two matrices, also does not hold for the product of two vectors. The minor product of two vectors is a .... while the major product of the two
(number/vector/matrix)
vectors is a .... It is true because the
(number/vector/matrix)
commutative law of multiplication does not hold for the product of two vectors.

----

number

matrix

---

63. The following minor and major products of two vectors

$$\begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix}$$

.... the same
(are/are not)

---

are not

---

64. Let $\underline{u}$ be a row vector and $\underline{v}$ be a column vector. Then

$\underline{u}\ \underline{v}$ .... equal to $\underline{v}\ \underline{u}$.
(is/is not)

---

is not

Unit XV: The Solution of Linear Equations

1. Let

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

and $A\underline{u} = \underline{v}$

Then

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

or

$$\begin{bmatrix} \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

and it can be written in the form of linear equations as

.... = ...

.... = ...

.... = ...

$$\begin{bmatrix} x + 4y + 3z \\ 2x + 5y + 4z \\ x - 3y - 2z \end{bmatrix}$$

$$\begin{aligned} x + 4y + 3z &= 1 \\ 2x + 5y + 4z &= 4 \\ x - 3y - 2z &= 5 \end{aligned}$$

2. Let

$$\begin{aligned} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + z &= 5 \end{aligned}$$

The above linear equations can be represented in the form of $A\underline{u} = \underline{v}$ as follows

$$\begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

3. Represent the following system of linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

in the form $A\underline{u} = \underline{v}$

$$\begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix} \begin{bmatrix} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{bmatrix} = \begin{bmatrix} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

4. We have seen that a system of linear equations can be represented in the form $A\underline{u} = \underline{v}$, where A is the coefficient matrix for the system of equations, and $\underline{u}$ and $\underline{v}$ are vectors. Also A and $\underline{u}$ are conformable for multiplication and the product $A\underline{u}$ is a ....
(vector/matrix)

---

vector

5. If A^{-1} is the inverse of the matrix A, then the equation $A\underline{u} = \underline{v}$ can be solved for the unknown vector $\underline{u}$, getting

$\underline{u} =$ ....

$A^{-1}\underline{v}$

6. Therefore a system of n linear equations for n unknowns may be written in terms of its coefficient matrix A as the equation $A\underline{u} = \underline{v}$ for the unknown vector $\underline{u}$. If A has an inverse matrix, then the solution is given by

$\underline{u} =$ ....

$A^{-1}\underline{v}$

7. The following system of linear equations

$$\begin{aligned} x + 4y + 3z &= 1 \\ 2x + 5y + 4z &= 4 \\ x - 3y - 2z &= 5 \end{aligned}$$

can be represented in the form of $A\underline{u} = \underline{v}$ as

$$\begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix} \begin{bmatrix} ..... \\ ..... \\ ..... \end{bmatrix} = \begin{bmatrix} ..... \\ ..... \\ ..... \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

---

8. If

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ 8 & -5 & 2 \\ -11 & 7 & -3 \end{bmatrix}$$

Therefore multiplying

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

both sides (on the left) by A^{-1} we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 8 & -5 & 2 \\ -11 & 7 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} ..... \\ ..... \\ ..... \end{bmatrix}$$

(after multiplication)

$$\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

---

9. If $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

Then the values of x, y and z are

x = ..., y = ..., and z = ...

3, -2, 2

---

10. Let

$$4x + 5z = 6$$
$$y - 6z = -2$$
$$3x + 4z = 3$$

The above system of linear equations can be represented in the form $A\underline{u}=\underline{v}$ as follows

$$\begin{bmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{bmatrix} \begin{bmatrix} ..... \\ ..... \\ ..... \end{bmatrix} = \begin{bmatrix} ..... \\ ..... \\ ..... \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 5 \\ 0 & 1 & -6 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

---

11. If the inverse of

$$\begin{bmatrix} 4 & 0 & 5 \\ 0 & 1 & -6 \\ 3 & 0 & 4 \end{bmatrix} \text{ is } \begin{bmatrix} 4 & 0 & -5 \\ -18 & 1 & 24 \\ -3 & 0 & 4 \end{bmatrix}$$

Then the equation

$$\begin{bmatrix} 4 & 0 & 5 \\ 0 & 1 & -6 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

can be solved for x, y and z getting

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix} \begin{bmatrix} \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \end{bmatrix} = \begin{bmatrix} \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & -5 \\ -18 & 1 & 24 \\ -3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -38 \\ -6 \end{bmatrix}$$

Unit XVI: The Theory of Linear Dependence

1. We learned in a previous unit that a determinant is zero if any of its rows (or columna) are linear combinations of other rows (or columns). The determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 7 \\ 5 & 15 & 11 \end{bmatrix}$$

is zero because the column 2 is equal to ...... times of column 1.

3
(three)

---

2. If in the matrix of the previous frame we denote the columns 1, 2 and 3 as vectors $\underline{v}_1$, $\underline{v}_2$ and $\underline{v}_3$ respectively, then:

(a) $\underline{v}_2 = k\,\underline{v}_1$ when k = ...

3

(b) but $\underline{v}_3 \neq$ ..... $\underline{v}_1$ whatever the value of k.

---

k

(c) When two vectors of the same size can be represented by a linear equation then they are called linearly dependent and if they cannot then they are called linearly independent. Therefore, in the previous frame $\underline{v}_1$ and $\underline{v}_2$ are .... and $\underline{v}_1$ and $\underline{v}_3$ are ....

---

linearly dependent

linearly independent

COMMENT

In the previous frame linear dependence and independence has been defined in terms of only two vectors. The general definition applies to any number of vectors. Thus in general, n vectors all of the same order, $\underline{v}_1$, $\underline{v}_2$, ..., $\underline{v}_n$ are said to be linearly dependent if non-zero constants k_1, k_2, ..., k_n exist such that

$$k_1\underline{v}_1 + k_2\underline{v}_2 + \ldots\ldots + k_n\underline{v}_n = 0.$$

If zeros are the only values of the k's for which this equation is true, the vectors are said to be linearly independent.

---

3. In the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 7 \\ 5 & 15 & 11 \end{bmatrix}$$

the columns 1 and 2 are linearly ...., and $|A|$ = ..., therefore A^{-1} does not exist.

---

dependent

zero

---

4. In the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & -1 \\ 5 & 9 & 11 \end{bmatrix}$$

the columns 1, 2 and 3 are .... and also $|A| \neq$ ....., therefore A^{-1} does exist.

----

linearly independent

zero

C O M M E N T

The property of the determinant which we noticed in the last two frames for a 3 x 3 matrix, is true in general: that if the rows (or columns) of a matrix are linearly dependent its determinant is zero and its inverse does not exist; conversely, if they are independent its determinant is non-zero and the inverse does exist. The minimum condition for the linear dependence of rows (or columns) of a matrix is that one of them be a linear combination of the others.

5. The number of independent rows in a matrix equals the number of independent columns and vice versa. The matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & -1 \\ 5 & 9 & 11 \end{bmatrix}$$

has 3 linearly independent rows therefore it has .... linearly independent columns also.

3

---

6. The rank of any matrix is the number of linearly independent rows (or columns) therein. The matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & -1 \\ 5 & 9 & 11 \end{bmatrix}$$

has 3 independent rows (as well as columns). Therefore the rank of the matrix A is ....

3

COMMENT

Let A be m x n matrix. If A contains at least one r-rowed minor determinant that does not vanish, but no non-vanishing (r+1)-rowed minor determinant, A is said to be of rank r. If A = 0, the rank is said to be zero. If r = m or r = n, A contains no (r + 1)-rowed minor determinant. If r<m and r<n, the definition implies that A contains at least one (r+1)-rowed minor determinant but that each such determinant vanishes.

---

7. From the previous comment it is clear that the rank of the matrix A of size m x n where m<n is equal to or less than ....

m

---

8. If A is a square matrix of order n and its rank r < n, then |A| = 0, and A^{-1} .... exist. Therefore the matrix A is singular.
(does or does not)

does not

---

9. If A is a square matrix of order n and its rank r=n, then $|A| \neq 0$, and A^{-1} .... exist. Therefore the matrix A is ....
(does or does not) (singular or non-singular)
and is called matrix of full rank.

does
non-singular

Note: The rank of the inverse of a matrix A of order n is also n, and $|A^{-1}| = 1/|A|$.

---

10. If A is a matrix of order mxn and has rank r, then A has r linearly

.... rows (columns).
(dependent or independent)

independent

---

11. To find out the rank of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & -1 \\ 5 & 9 & 11 \end{bmatrix}$$

(a) the first row is multiplied by -2 and -5, and then added to the 2nd and 3rd rows respectively to make the sub-diagonal elements of rows 2 and 3 in column 1, equal to zero; we get

$$A_1 = \begin{bmatrix} 1 & 3 & 4 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -9 \\ 0 & -6 & -9 \end{bmatrix}$$

(b) The second row of A_1 is multiplied by 6 and then added to the 3rd row of A_1 to make the sub-diagonal element of row 3 in column 2, equal to zero; we get

$$A_2 = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -9 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 9 \\ 0 & 0 & -63 \end{bmatrix}$$

(c) In A_2 we have made all the sub-diagonal elements of A equal to zero. Now the number of non-zero diagonal elements of A_2 is the rank of the matrix A. Therefore the rank of A is ....

3

---

12. To find out the rank of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 8 & 6 & 3 & 5 \\ 3 & -1 & 0 & 1 \end{bmatrix}$$

(a) the first row is multiplied by .... and ...., and then added to the 2nd and 3rd row respectively to make the sub-diagonal elements of rows 2 and 3 in column 1, equal to zero; we get

(continued on next page)

$$A_1 = \begin{bmatrix} 1 & 4 & 2 & 3 \\ \text{.....} & \text{.....} & \text{.....} & \text{.....} \\ \text{.....} & \text{.....} & \text{.....} & \text{.....} \end{bmatrix}$$

-8 and -3

$$A_1 = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -26 & -13 & -19 \\ 0 & -13 & -6 & -8 \end{bmatrix}$$

(b) the second row of A_1 is multiplied by -1 and 3rd row by .... and then added to the 3rd row of A_1 to make the sub-diagonal element of row 3 in column 2 equal to zero; we get

$$A_2 = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -26 & -13 & -19 \\ \text{.....} & \text{.....} & \text{.....} & \text{.....} \end{bmatrix}$$

2

$$A_2 = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -26 & -13 & -19 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(c) Now the number of non-zero diagonal elements of A_2 is .... and therefore the rank of A is ....

---

3 ; 3

COMMENT

The method of finding the rank of a matrix, which we have used in the previous frames is quite general: initially the first row is used to reduce the sub-diagonal elements of the first column to zero, basing the calculations on the first term of the first row. Then the elements below the diagonal in the second column are reduced to zeros using the second row, based on its diagonal term. Because the sub-diagonal elements of the first column have already been made zero in the first step of this procedure this second step does not effect their values. And so the process is continued, using each row this way in turn, until all remaining rows are zero or until the last row is reached. The number of non-zero diagonal elements is then the rank.

Unit XVII: The Characteristic Equation of a Matrix

1. Let

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix}$$

be a square matrix of order 2 and k is a scalar variable then

$$A\text{-}kI = \begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix} -k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 3\text{-}k & 5 \\ 1 & 7\text{-}k \end{bmatrix}$$

---

2. Let B be an n-square matrix and λ a scalar variable, then

(a) the matrix B-λ I is called the characteristic matrix of B. Therefore in the previous frame the characteristic matrix of A is

$$\begin{bmatrix} \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 3\text{-}k & 5 \\ 1 & 7\text{-}k \end{bmatrix}$$

(b) the determinant $|B-\lambda I| = f(\lambda)$ is called the characteristic determinant of B; the equation $f(\lambda) = 0$ is called the characteristic equation of B. From frame 1 we have

$$|A-kI| = \begin{vmatrix} 3-k & 5 \\ 1 & 7-k \end{vmatrix} = (3-k)\ (7-k) - 5 = k^2 - 10k + 16,$$

therefore $k^2 - 10k + 16 = 0$ is the .... of A.

---

characteristic equation

(c) the roots of $f(\lambda) = 0$ are called the characteristic roots, the latent roots, the characteristic values , or the eigenvalues of B. Solving the characteristic equation of A, $k^2-10k+16 = 0$, $k = 8$ or 2. Therefore the characteristic roots of A are .... and ....

---

8 and 2

---

3. Let

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \qquad \text{and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) The characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 4 & 1 \\ 2 & 1-\lambda & 0 \\ -1 & 3 & 1-\lambda \end{bmatrix}$$

(b) The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} = 0$$

or $\lambda(\lambda^2 - 3\lambda - 4) = 0$

- -

$$\begin{vmatrix} 1-\lambda & 4 & 1 \\ 2 & 1-\lambda & 0 \\ -1 & 3 & 1-\lambda \end{vmatrix} = 0$$

(c) The characteristic equation of A is

$$\lambda(\lambda^2 - 3\lambda - 4) = 0 \quad \text{or} \quad \lambda(\lambda + 1)(\lambda - 4) = 0$$

Therefore the characteristic roots of A are ....., ... and ..... .

- -

0, -1, 4

---

4. Let

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -2 & -2 & -1 \end{bmatrix}$$

(a) The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} ..... & ..... & ..... \\ ..... & ..... & ..... \\ ..... & ..... & ..... \end{vmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & 4 & -2 \\ 4 & 2-\lambda & -2 \\ -2 & -2 & -1-\lambda \end{vmatrix} = 0$$

(b) The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 4 & -2 \\ 4 & 2-\lambda & -2 \\ -2 & -2 & -1-\lambda \end{vmatrix} = 0$$

which after expanding the determinant, becomes

$$-\lambda^3 + 3\lambda^2 + 24\lambda + 28 = 0$$

or $(\lambda + 2)^2 (7 - \lambda) = 0$

Therefore the characteristic roots of A are ....., ... and ..... .

-2, -2, 7

---

5. Let

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix}$$

and its latent roots are 2 and 8. Let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors

such that

$$A\underline{u} = 2\underline{u} \quad \text{and} \quad A\underline{v} = 8\underline{v}$$

or $$\begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 2\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 8\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

or $$\begin{bmatrix} 3u_1 + 5u_2 \\ \ldots\ldots\ldots \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 2u_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \end{bmatrix} = \begin{bmatrix} 8v_1 \\ 8v_2 \end{bmatrix}$$

$$\begin{bmatrix} 3u_1 + 5u_2 \\ u_1 + 7u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 2u_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3v_1 + 5v_2 \\ v_1 + 7v_2 \end{bmatrix} = \begin{bmatrix} 8v_1 \\ 8v_2 \end{bmatrix}$$

---

6. From the previous frame we have

$$\begin{bmatrix} 3u_1 + 5u_2 \\ u_1 + 7u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 2u_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3v_1 + 5v_2 \\ v_1 + 7v_2 \end{bmatrix} = \begin{bmatrix} 8v_1 \\ 8v_2 \end{bmatrix}$$

Taking the arbitrary values of $u_1 = -5$ and $v_1 = 1$, and then solving the equations $3u_1 + 5u_2 = 2u_1$ and $3v_1 + 5v_2 = 8v_1$, we get

$u_2 = \ldots\ldots$ and $v_2 = \ldots\ldots$

$u_2 = 1$, $v_2 = 1$

---

7. From the frames 5 and 6 we can see that corresponding to the latent roots 2 and 8 of the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix}$, there are two vectors

$$\underline{u} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} \quad \text{and} \quad \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{such that}$$

$$A\underline{u} = 2\underline{u} \qquad \text{and} \quad A\underline{v} = 8\underline{v}$$

i.e. $\begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} -5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Here $\underline{u}$ and $\underline{v}$ are the latent vectors of A corresponding to the latent roots (2 and 8) of A.

---

8. Let A be a matrix of order n and therefore the characteristic equation $|A-\lambda I| = 0$ is a polynomial in λ of degree n. If it has all the solutions then:

(a) How many latent roots of A do you expect? ....

n

(b) How many latent vectors of A do you expect? ....

n

(c) If λ_i and $\underline{u}_i$ (for i = 1, 2, ..., n) are the latent roots and vectors of A, then

$A\underline{u}_i$ = ...

$\lambda_i\underline{u}_i$

9. Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & -2 \\ 4 & 1 & 0 \end{bmatrix}$$

(a) The sum of the diagonal elements (which is called the trace) of A = ... + ... + ... = 5

2 + 3 + 0

(b) The sum of the minors of order 2 of the diagonal elements (which are called the principal minors of order 2) 2, 3 and 0 of $|A|$

$$= \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{vmatrix} = \dots\dots - 4 + 9 = 7$$

$$\begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix} \; ; \; 2$$

(c) The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ 3 & 3-\lambda & -2 \\ 4 & 1 & -\lambda \end{vmatrix} = 0$$

and it can be expanded by diagonal elements (called the diagonal expansion).

It is

$$(-\lambda)^3+(-\lambda)^2(\text{trace of A})+(-\lambda)\ (\text{sum of the second order minors of A})+|A| = 0$$

or $-\lambda^3 + \lambda^2\ (\ldots\ldots) - \lambda\ (\ 7\ ) + 3 = 0$

- -

5

(d) The roots of the characteristic equation

$$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

or $(\lambda-1)^2\ (\lambda-3) = 0$

are $\lambda_1 = \ldots\ldots$, $\lambda_2 = \ldots\ldots$, and $\lambda_3 = \ldots\ldots\ldots$.

- -

1, 1, and 3

COMMENT

The method of diagonal expansion of the characteristic equation in the previous frame is quite general and can be used for a matrix of order n.

Unit XVIII: Special Types of Matrices

1. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

(a) The elements of the first row are ....., ..., ... and the elements of the first column are ....., ..., ...

---

a_{11}, a_{12}, a_{13}

a_{11}, a_{12}, a_{13}

(b) The elements of the second row are ....., ..., ... and the elements of the second column are ....., ..., ...

---

a_{12}, a_{22}, a_{23}

a_{12}, a_{22}, a_{23}

(c) The elements of the 3rd row are ....., ..., ... and of the 3rd column ....., ..., ...

---

a_{13}, a_{23}, a_{33}

a_{13}, a_{23}, a_{33}

(d) The transpose of A is

$$A' = \begin{bmatrix} \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix} = \ldots\ldots$$

(name of the matrix)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = A$$

---

2. Now let

$$B = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} ; \quad \text{then}$$

(a) The elements of the 1st row are ....., ..., ...

and the elements of the 1st column are ....., ..., ...

$0, \quad a_{12}, \quad a_{13}$

$0, \quad -a_{12}, \quad -a_{13}$

(b) The elements of the second row are ....., ..., ...

and of the second column ....., ..., ...

$-a_{12}$, 0, a_{23}

a_{12}, 0, $-a_{23}$

(c) The elements of the 3rd row are ....., ..., ...

and of the 3rd column ....., ..., ...

- -

$-a_{13}$, $-a_{23}$, 0

a_{13}, a_{23}, 0

(d) The transpose of B is

$$B' = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} = - \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} = -\dots$$

- -

$$\begin{bmatrix} 0 & -a_{12} & -a_{13} \\ a_{12} & 0 & -a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix} = - \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} = -B$$

COMMENT

We have seen in frame 1 that the rows of A are equal to the corresponding columns of A, and also $A = A'$, therefore A is a symmetric matrix.

In frame 2, the rows of B are equal to minus times the corresponding columns of B, and also $B' = -B$, therefore B is a skew-symmetric matrix.

In general a matrix A is said to be symmetric if it is equal to its transpose, i.e., if $A = A'$ or $a_{ij} = a_{ji}$; $(i,j = 1, 2, \ldots\ldots, n)$. A is said to be skew-symmetric if it is equal to the negative of its transpose, i.e., $A = -A'$, or $a_{ij} = -a_{ji}$, $a_{ii} = 0$; $(i,j = 1, 2, \ldots\ldots, n)$.

3. If A is such a matrix that its inverse equals its transpose, i.e. $A^{-1} = A'$, then A is said to be an orthogonal matrix.

Also $AA^{-1} = AA'$

or $\ldots\ldots = AA'$

(multiplication of A and A^{-1})

I

4. Let A be an orthogonal matrix and λ a latent root, then

$$|A - \lambda I| = 0.$$

Multiplying throughout by A' we get

$$|AA' - \lambda IA'| = 0$$

or $|\ldots\ldots - \lambda A'| = 0$

Multiplying throughout by $\frac{1}{\lambda}$ we get

$$\left|\frac{1}{\lambda} I - \ldots\ldots\right| = 0$$

or $\left|A - \frac{1}{\lambda} I\right| = 0$ because $A = A'$

It shows that if λ is a latent root of an orthogonal matrix then so is $1/\lambda$.

I

A'

---

5. Let A be an orthogonal matrix so that $AA' = I$. Then

$$|AA'| = |I| = +1 \text{ or } -1$$

Therefore the determinant of A is ..... or ..... .

+1 or -1

---

6. Let

$$A = \begin{bmatrix} 3 & -6 & -4 \\ -6 & 4 & 2 \\ -4 & 2 & -1 \end{bmatrix}$$

be a symmetric matrix with its latent roots $\lambda = -1, -4, 11$.

Corresponding to the latent roots, the latent vectors are

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

which can be represented as columns of a matrix P,

$$\text{i.e.} \quad P = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

(a) P can be normalized dividing elements of each column by the square root of the sum of the squares of the elements in the column. Therefore normalizing column 1 of P, elements 1, 2 and -2 should be divided by $\sqrt{1^2 + 2^2 + 2^2} = 3$; normalizing column 2, elements of 2, 1 and 2 should be divided by $\sqrt{.... + ... + ...} = 3$; and normalizing column 3, elements 2, -2, -1 should be divided by $\sqrt{.... + ... + ...} = ...$.

$$\sqrt{2^2 + 1^2 + 2^2}$$

$$\sqrt{2^2 + 2^2 + 1^2} = 3$$

(b) Normalized form of P is obtained by dividing columns 1, 2 and 3 by 3.

Therefore normalized P is

$$P^{*} = \begin{bmatrix} \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix} = 1/3 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/2 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}$$

(c) The multiplication of the transpose of P^{*} to P^{*} is

$$P'^{*}\, P^{*} = 1/9 \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} = 1/9 \begin{bmatrix} \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix} = \cdots\cdots$$

Therefore P^{*} is an orthogonal matrix.

$$1/9 \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I$$

(d)

$$P'^{*}\,A\,P^{*} = 1/9\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}\begin{bmatrix}3 & -6 & -4\\ -6 & 4 & 2\\ -4 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}$$

$$= 1/9\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}\begin{bmatrix}\text{....} & \text{....} & \text{....}\\ \text{....} & \text{....} & \text{....}\\ \text{....} & \text{....} & \text{....}\end{bmatrix} = 1/9\begin{bmatrix}\text{....} & \text{....} & \text{....}\\ \text{....} & \text{....} & \text{....}\\ \text{....} & \text{....} & \text{....}\end{bmatrix} = D$$

where D is the diagonal matrix of latent roots.

$$1/9\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}\begin{bmatrix}-1 & -8 & 22\\ -2 & -4 & -22\\ 2 & -8 & -11\end{bmatrix} = 1/9\begin{bmatrix}-9 & 0 & 0\\ 0 & -36 & 0\\ 0 & 0 & 99\end{bmatrix}$$

COMMENT

The results of the previous frame are true for general. If A is an n-square symmetric matrix whose characteristic roots are $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$, there exists an orthogonal matrix P^{*} such that $P'^{*}\,A\,P^{*} = D$ where D is the diagonal matrix of the latent roots.

---

7. A square matrix which has only zeros above (or below) the main diagonal is called a triangular matrix.

The matrix

$$A = \begin{bmatrix}1 & 3 & 5\\ 0 & 2 & 7\\ 0 & 0 & 3\end{bmatrix}$$

is a .... matrix.

---

triangular
